Artificial Intelligence

Propositional logic: inference algorithms

Outline

- Propositional (Boolean) logic
- Equivalence, validity, satisfiability
- Inference rules and theorem proving
  - forward chaining
  - backward chaining
  - resolution

Propositional logic: Syntax

- Propositional logic is the simplest logic – illustrates basic ideas
- The proposition symbols P_1, P_2 etc are sentences
  - If S is a sentence, $\neg$S is a sentence (negation)
  - If S_1 and S_2 are sentences, $S_1 \wedge S_2$ is a sentence (conjunction)
  - If S_1 and S_2 are sentences, $S_1 \vee S_2$ is a sentence (disjunction)
  - If S_1 and S_2 are sentences, $S_1 \Rightarrow S_2$ is a sentence (implication)
  - If S_1 and S_2 are sentences, $S_1 \Leftrightarrow S_2$ is a sentence (biconditional)

Truth tables for connectives

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
false	*false*	*true*	*false*	*false*	*true*	*true*
false	*true*	*true*	*false*	*true*	*true*	*false*
true	*false*	*false*	*false*	*true*	*false*	*false*
true	*true*	*false*	*true*	*true*	*true*	*true*

Logical equivalence

- Two sentences are logically equivalent iff true in same models: α ≡ ß iff α ╞ β and β ╞ α

$$
\begin{aligned}
(\alpha \wedge \beta) &\equiv (\beta \wedge \alpha) && \text{commutativity of } \wedge \\
(\alpha \vee \beta) &\equiv (\beta \vee \alpha) && \text{commutativity of } \vee \\
((\alpha \wedge \beta) \wedge \gamma) &\equiv (\alpha \wedge (\beta \wedge \gamma)) && \text{associativity of } \wedge \\
((\alpha \vee \beta) \vee \gamma) &\equiv (\alpha \vee (\beta \vee \gamma)) && \text{associativity of } \vee \\
\neg(\neg\alpha) &\equiv \alpha && \text{double-negation elimination} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\beta \Rightarrow \neg\alpha) && \text{contraposition} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\alpha \vee \beta) && \text{implication elimination} \\
(\alpha \Leftrightarrow \beta) &\equiv ((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)) && \text{biconditional elimination} \\
\neg(\alpha \wedge \beta) &\equiv (\neg\alpha \vee \neg\beta) && \text{de Morgan} \\
\neg(\alpha \vee \beta) &\equiv (\neg\alpha \wedge \neg\beta) && \text{de Morgan} \\
(\alpha \wedge (\beta \vee \gamma)) &\equiv ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma)) && \text{distributivity of } \wedge \text{ over } \vee \\
(\alpha \vee (\beta \wedge \gamma)) &\equiv ((\alpha \vee \beta) \wedge (\alpha \vee \gamma)) && \text{distributivity of } \vee \text{ over } \wedge
\end{aligned}
$$

Validity and satisfiability

A sentence is valid if it is true in all models,

e.g., *True*, $A \vee \neg A$, $A \Rightarrow A$, $(A \wedge (A \Rightarrow B)) \Rightarrow B$

Validity is connected to inference via the Deduction Theorem:

$KB \models \alpha$ if and only if $(KB \Rightarrow \alpha)$ is valid

A sentence is satisfiable if it is true in some model

e.g., $A \vee B$, C

A sentence is unsatisfiable if it is true in no models

e.g., $A \wedge \neg A$

Satisfiability is connected to inference via the following:

$KB \models \alpha$ if and only if $(KB \wedge \neg\alpha)$ is unsatisfiable

Example: Wumpus world KB

- Let $P_{i,j}$ be true if there is a pit in [i, j].
- Let $B_{i,j}$ be true if there is a breeze in [i, j].
- Pits cause breezes in adjacent squares
- The KB has 5 sentences:

$\neg P_{1,1}$

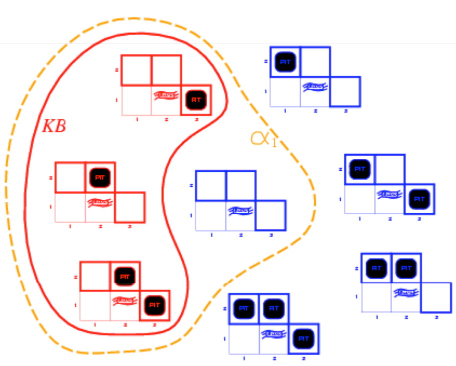
$\neg B_{1,1}$

$B_{2,1}$

$B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$

$B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$

Wumpus models



- *KB* = wumpus-world rules + observations
- α_1 = "[1,2] is safe", $KB \models \alpha_1$, proved by model checking

Truth tables for inference

$B_{1,1}$	$B_{2,1}$	$P_{1,1}$	$P_{1,2}$	$P_{2,1}$	$P_{2,2}$	$P_{3,1}$	KB	α_1
false	*false*	*false*	*false*	*false*	*false*	*false*	*false*	*true*
false	*false*	*false*	*false*	*false*	*false*	*true*	*false*	*true*
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
false	*true*	*false*	*false*	*false*	*false*	*false*	*false*	*true*
false	*true*	*false*	*false*	*false*	*false*	*true*	<u>*true*</u>	<u>*true*</u>
false	*true*	*false*	*false*	*false*	*true*	*false*	<u>*true*</u>	<u>*true*</u>
false	*true*	*false*	*false*	*false*	*true*	*true*	<u>*true*</u>	<u>*true*</u>
false	*true*	*false*	*false*	*true*	*false*	*false*	*false*	*true*
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
true	*true*	*true*	*true*	*true*	*true*	*true*	*false*	*false*

Inference by enumeration

```
function TT-ENTAILS?(KB, α) returns true or false
   symbols ← a list of the proposition symbols in KB and α
   return TT-CHECK-ALL(KB, α, symbols, [])

function TT-CHECK-ALL(KB, α, symbols, model) returns true or false
   if EMPTY?(symbols) then
      if PL-TRUE?(KB, model) then return PL-TRUE?(α, model)
      else return true
   else do
      P ← FIRST(symbols); rest ← REST(symbols)
      return TT-CHECK-ALL(KB, α, rest, EXTEND(P, true, model)) and
               TT-CHECK-ALL(KB, α, rest, EXTEND(P, false, model))
```

- For *n* symbols, time complexity is $O(2^n)$, space complexity is $O(n)$

Proof methods

- Proof methods divide into (roughly) two kinds:
  - Application of inference rules
    - Legitimate (sound) generation of new sentences from old
    - Proof = a sequence of inference rule applications
      Can use inference rules as operators in a standard search algorithm
    - Typically require transformation of sentences into a normal form
  - Model checking
    - truth table enumeration (always exponential in *n*)
    - improved backtracking, e.g., Davis--Putnam-Logemann-Loveland (DPLL) algorithm
    - heuristic search in model space (sound but incomplete)

Inference rule: Resolution

- Conjunctive Normal Form (CNF) : conjunction of disjunctions of literals

  E.g., $(A \lor \neg B) \land (B \lor \neg C \lor \neg D)$
- Resolution inference rule (for CNF):

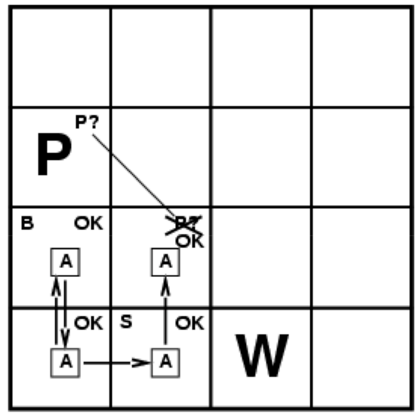
$$\frac{\ell_i \lor \ldots \lor \ell_k, \qquad m_1 \lor \ldots \lor m_n}{\ell_i \lor \ldots \lor \ell_{i-1} \lor \ell_{i+1} \lor \ldots \lor \ell_k \lor m_1 \lor \ldots \lor m_{j-1} \lor m_{j+1} \lor \ldots \lor m_n}$$

where ℓ_i and m_j are complementary literals.

- Resolution is sound and complete for propositional logic

Resolution example

$$\frac{P_{1,3} \lor P_{2,2}, \quad \neg P_{2,2}}{P_{1,3}}$$



Conversion to CNF

Convert $B_{1,1} \Leftrightarrow (P_{1,2} \lor P_{2,1})$ into CNF:

1. Eliminate $\Leftrightarrow$, replacing $\alpha \Leftrightarrow \beta$ with $(\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha)$.
   $(B_{1,1} \Rightarrow (P_{1,2} \lor P_{2,1})) \land ((P_{1,2} \lor P_{2,1}) \Rightarrow B_{1,1})$
2. Eliminate $\Rightarrow$, replacing $\alpha \Rightarrow \beta$ with $\neg\alpha \lor \beta$.
   $(\neg B_{1,1} \lor P_{1,2} \lor P_{2,1}) \land (\neg(P_{1,2} \lor P_{2,1}) \lor B_{1,1})$
3. Move $\neg$ inwards using de Morgan's rules and double-negation:
   $(\neg B_{1,1} \lor P_{1,2} \lor P_{2,1}) \land ((\neg P_{1,2} \lor \neg P_{2,1}) \lor B_{1,1})$
4. Apply distributivity law ($\land$ over $\lor$) and flatten:
   $(\neg B_{1,1} \lor P_{1,2} \lor P_{2,1}) \land (\neg P_{1,2} \lor B_{1,1}) \land (\neg P_{2,1} \lor B_{1,1})$

Resolution algorithm

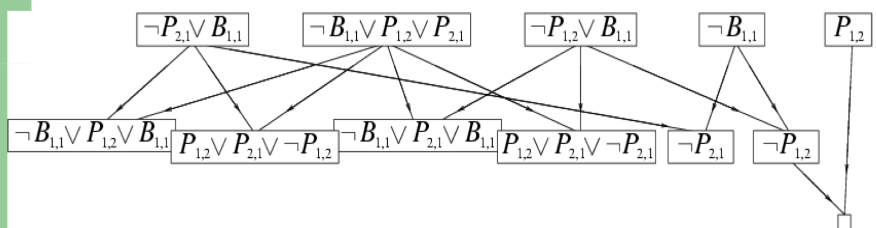
- Proof by contradiction: To show $KB \models \alpha$, we show that $KB \wedge \neg\alpha$ is unsatisfiable
- First, $KB \wedge \neg\alpha$ is converted into CNF. Then, the resolution rule is applied to the resulting clauses. Each pair that contains complementary literals is resolved to produce a new clause, which is added to the set if it is not already present. The process continues until one of two things happens:
  - there are no new clauses that can be added, in which case *KB* does not entail α; or,
  - Two clauses resolve to yield the empty clause, in which case *KB* entails α.

Resolution algorithm

```
function PL-RESOLUTION(KB, α) returns true or false
   clauses ← the set of clauses in the CNF representation of KB ∧ ¬α
   new ← { }
   loop do
      for each Ci, Cj in clauses do
         resolvents ← PL-RESOLVE(Ci, Cj)
         if resolvents contains the empty clause then return true
         new ← new ∪ resolvents
      if new ⊆ clauses then return false
      clauses ← clauses ∪ new
```

Resolution example

- *KB* = $(B_{1,1} \Leftrightarrow (P_{1,2} \lor P_{2,1})) \land \neg B_{1,1}$
- $\alpha = \neg P_{1,2}$



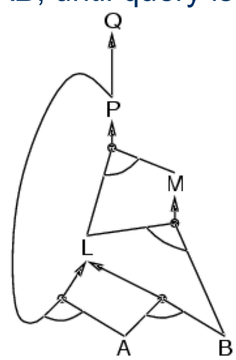
Horn clauses

- Real-world knowledge bases often contain only clauses of a restricted kind called Horn clauses.
- Horn clause =
  - proposition symbol (fact); or
  - (conjunction of symbols) (called body) $\Rightarrow$ symbol (called head)
  - E.g., $C \land D \Rightarrow B$
- Can be used with forward chaining or backward chaining.
- These algorithms are very natural and run in linear time

Forward chaining

- Idea: fire any rule whose premises are satisfied in the *KB*, add its conclusion to the *KB*, until query is found

$$P \Rightarrow Q$$
$$L \wedge M \Rightarrow P$$
$$B \wedge L \Rightarrow M$$
$$A \wedge P \Rightarrow L$$
$$A \wedge B \Rightarrow L$$
$$A$$
$$B$$

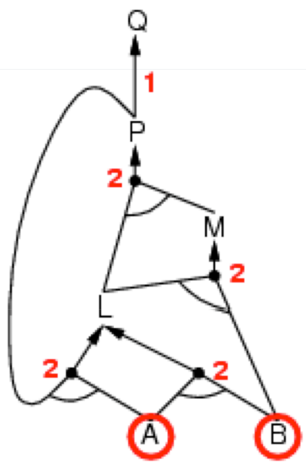


Forward chaining algorithm

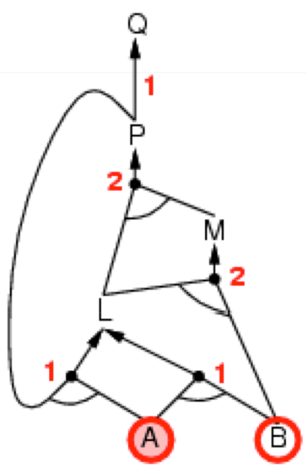
```
function PL-FC-ENTAILS?(KB, q) returns true or false
  local variables: count, a table, indexed by clause, initially the number of premises
                   inferred, a table, indexed by symbol, each entry initially false
                   agenda, a list of symbols, initially the symbols known to be true

  while agenda is not empty do
      p ← POP(agenda)
      unless inferred[p] do
          inferred[p] ← true
          for each Horn clause c in whose premise p appears do
              decrement count[c]
              if count[c] = 0 then do
                  if HEAD[c] = q then return true
                  PUSH(HEAD[c], agenda)
  return false
```

- Forward chaining is sound and complete for Horn KB

Forward chaining example



Forward chaining example



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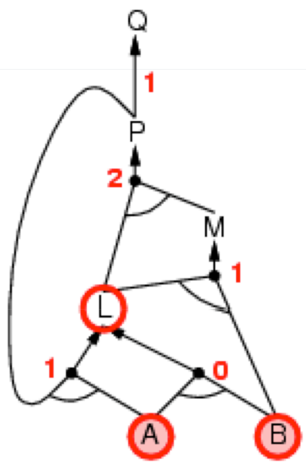

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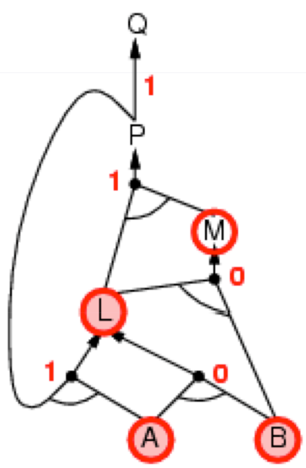

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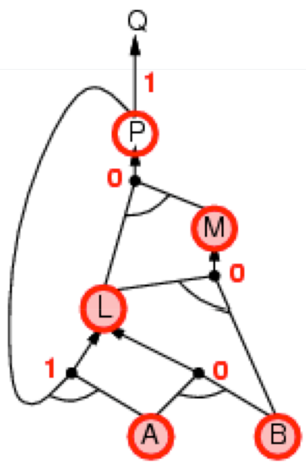

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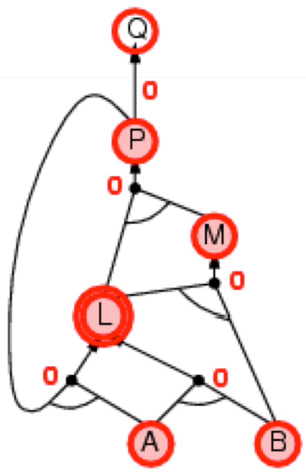
Forward chaining example



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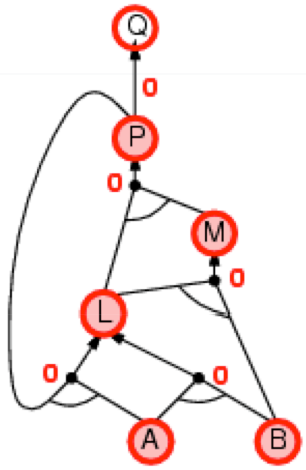
Forward chaining example



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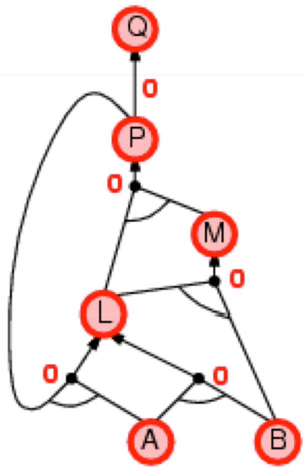

Proof of completeness

- FC derives every atomic sentence that is entailed by *KB*
  1. FC reaches a fixed point where no new atomic sentences are derived
  2. Consider the final state as a model *m*, assigning true/false to symbols
  3. Every clause in the original *KB* is true in *m*

     $a_1 \wedge \ldots \wedge a_k \Rightarrow b$
  4. Hence *m* is a model of *KB*
  5. If $KB \models q$, *q* is true in every model of *KB*, including *m*

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Backward chaining

Idea: work backwards from the query *q*:

to prove *q* by BC,

check if *q* is known already, or

prove by BC all premises of some rule concluding *q*

Avoid loops: check if new subgoal is already on the goal stack

Avoid repeated work: check if new subgoal

1. has already been proved true, or
2. has already failed

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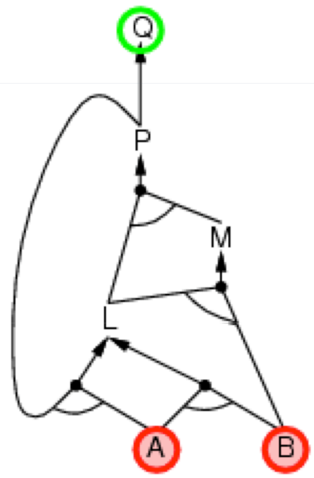
Backward chaining example



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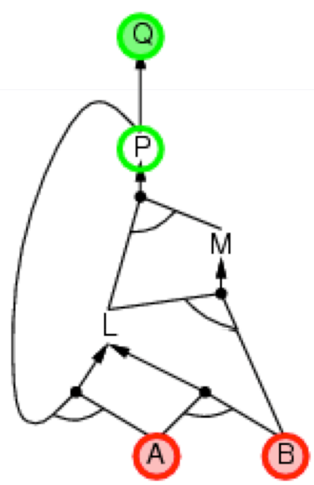

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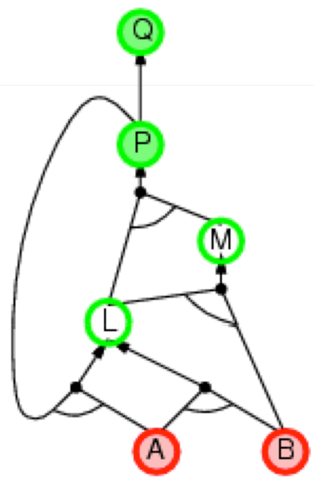

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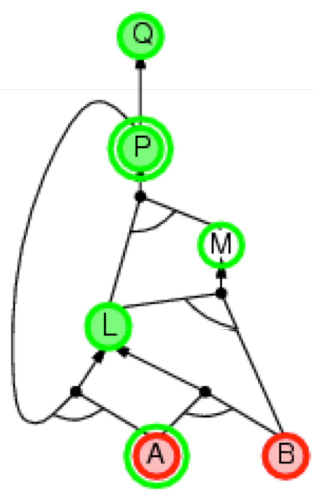

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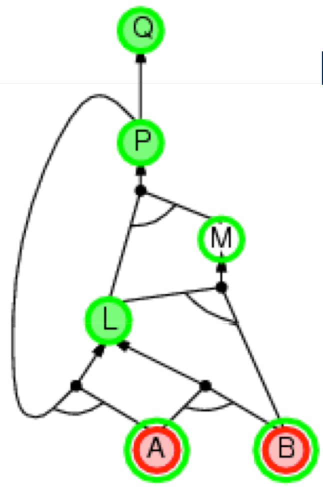

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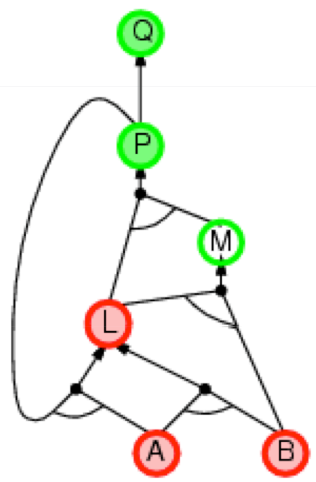

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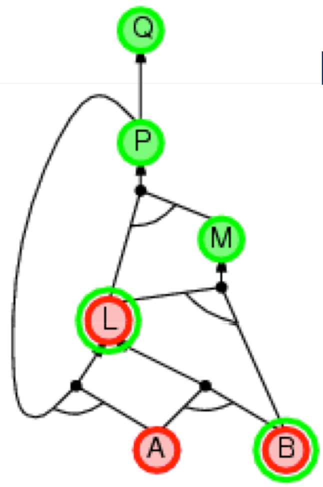

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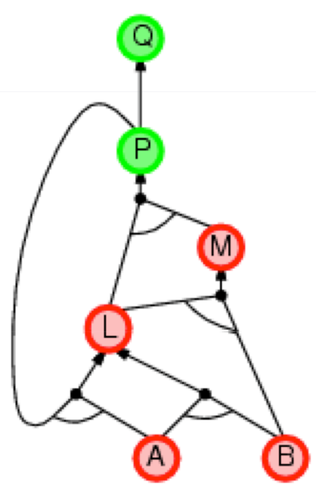

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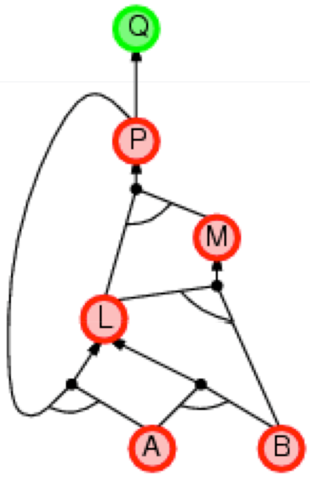

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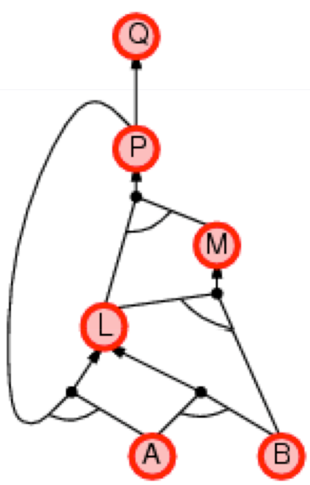

Forward vs. backward chaining

- FC is data-driven reasoning. It can be used within an agent to derive conclusions from incoming percepts, often without a specific query in mind.
- FC may do lots of work that is irrelevant to the goal
- BC is goal-driven, appropriate for problem-solving,
  - e.g., Where are my keys? What shall I do now?
- Complexity of BC can be much less than linear in size of KB

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Expressiveness limitation of propositional logic

- In Wumpus world, KB must contain sentences for every single square, not “general rules”
- Propositional logic does not scale to environments of unbounded size because it lacks the expressive power to deal concisely with time, space, and universal patterns of relationships among objects.

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Summary

- Logical agents apply inference to a knowledge base to derive new information and make decisions
- Basic concepts of logic:
  - syntax: formal structure of sentences
  - semantics: truth of sentences wrt models
  - entailment: necessary truth of one sentence given another
  - inference: deriving sentences from other sentences
  - soundness: derivations produce only entailed sentences
  - completeness: derivations can produce all entailed sentences
- Resolution is complete for propositional logic
- Forward, backward chaining are linear-time, complete for Horn clauses
- Propositional logic lacks expressive power