BIPARTITE-ASSEMBLY

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Abstract: Let A and B be nonempty sets of positive integers. We study the problem of finding bipartite graphs $G$ with bipartition sets X and Y such that every element in A is the degree of at least one vertex in X and every element of B is the degree of at least one vertex in Y. In addition to the question of existence the problem of determining the minimum order and size of graphs that are realizable for a given A and B is considered.

1. Introduction

Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ be nonempty sets of positive integers with the elements in each set written in ascending magnitude; that is, $a_1 < a_2 < \ldots < a_m$ and $b_1 < b_2 < \ldots < b_n$.

Problem 1: Find bipartite graphs $G$ with bipartition sets X and Y such that the degrees of the vertices in X belong to A and the degrees of the vertices in Y belong to B. Additionally, it is required that each element of A (B) is the degree of some vertex in X (Y).

Call a bipartite graph that is a solution to Problem 1 an $(A, B)$-graph. An edge in an $(A, B)$-graph $G$ is denoted $(x, y)$ where $x \in X$ and $y \in Y$.

Throughout this paper $\sum_{i=1}^{m} a_i$ and $\sum_{j=1}^{d} b_j$ will be denoted $s$ and $t$, respectively.

Theorem 1.1. For all $(A, B)$, an $(A, B)$-graph exists if and only if a $(B, A)$-graph exists.

Proof. Let $G$ be an $(A, B)$-graph of order $w$. Since $G$ is bipartite X can be canonically labeled 1, 2, ..., $x$ and Y labeled $x + 1$, $x + 2$, ..., $w$. Let each vertex $i$ of $G$ be relabeled $w - i + 1$ to obtain the graph $G^\#$. Then $G^\#$ is isomorphic to $G$, $X^\# = \{1, 2, \ldots, w - x\}$, $Y^\# = \{w - x + 1, w - x + 2, \ldots, w\}$, and $X^\#$ and $Y^\#$ correspond to B and A, respectively, that is, $G^\#$ is a $(B, A)$-graph.
Theorem 1.2. Problem 1 has a solution for all pairs \((A, B)\), with realization; an \((A, B)\)-graph of order \(ns + mt\) and size \(st\).

**Proof.** Let \(A = \{a_1, a_2, \ldots, a_m\}\) and \(B = \{b_1, b_2, \ldots, b_n\}\). Define \(H_{i,j}\) as the \((\{a_i\}, \{b_j\})\)-graph with \(b_j\) vertices in \(X\), \(a_i\) vertices in \(Y\), with each edge \((x, y)\) having \(\text{deg}(x) = a_i\) and \(\text{deg}(y) = b_j\). Then, \(H_{i,j}\) has order \(a_i + b_j\) and size \(a_i b_j\). In particular, \(H_{i,j}\) is isomorphic with the complete bipartite graph \(K_{b_j, a_i}\).

Then \(G = \bigcup_{i,j} H_{i,j}\), where the union is taken over all pairs \(i, j\) \(1 \leq i \leq m, 1 \leq j \leq n\) is clearly, an \((A, B)\)-graph of order \(\sum_{i=1}^{m} a_i + \sum_{j=1}^{n} b_j = ns + mt\) and size \(\sum_{i=1}^{m} a_i \left(\sum_{j=1}^{n} b_j\right) = \sum_{j=1}^{n} b_j \left(\sum_{i=1}^{m} a_i\right) = st\). 

Note: \(G\) is not connected using this construction except when \(n = m = 1\).

A solution of Problem 1 with a smaller order realization than that given by Theorem 1.2 is provided through the following theorem.

Theorem 1.3. Problem 1 has a solution for all pairs \((A, B)\) with a realization an \((A, B)\)-graph of order and size, respectively,

(a) \(s + t\) and \(\sum_{i=1}^{m} a_i b_j\), when \(m = n\),

(b) \(s + t + (n-m)a_i\) and \(\sum_{i=1}^{m} a_i b_j + \sum_{j=m+1}^{n} a_i b_j\), when \(m < n\),

(c) \(s + t + (m-n)b_i\) and \(\sum_{i=1}^{m} a_i b_j + \sum_{j=n+1}^{m} a_i b_j\), when \(m > n\).

**Proof.** Let \(A = \{a_1, a_2, \ldots, a_m\}\) and \(B = \{b_1, b_2, \ldots, b_n\}\). As in Theorem 1.2, define \(H_{i,j}\) as the \((\{a_i\}, \{b_j\})\)-graph with \(b_j\) vertices in \(X\), \(a_i\) vertices in \(Y\), with each edge \((x, y)\) having \(\text{deg}(x) = a_i\) and \(\text{deg}(y) = b_j\). Then, \(G' = \bigcup_{i,j} H_{i,j}\), with the \(i, j\) defined as follows

(a) if \(m = n\), then \(i = j\) for \(1 \leq i \leq m\),

(b) if \(m < n\), then \(i = j\) for \(1 \leq i \leq m\); and \(i = 1\) for \(j = m + 1, m + 2, \ldots, n\),

(c) if \(m > n\), then \(i = j\) for \(1 \leq j \leq n\); and \(j = 1\) for \(i = n + 1, n + 2, \ldots, m\)

is an \((A, B)\)-graph with order

\[
\sum_{i=1}^{m} a_i + \sum_{j=1}^{n} b_j = s + t \quad \text{in Case (a)},
\sum_{i=1}^{m} a_i + \sum_{j=m+1}^{n} b_j + (n-m)a_i = s + t + (n-m)a_i \quad \text{in Case (b)},
\sum_{i=1}^{m} a_i + \sum_{j=n+1}^{m} b_j + (m-n)b_i = s + t + (m-n)b_i \quad \text{in Case (c)}.
\]

The size of
\[ G \text{ is } \sum_{i=1}^{m} a_i b_i, \sum_{i=1}^{n} a_i b_j + \sum_{j=m+1}^{n} a_j b_j, \text{ and } \sum_{i=1}^{m} a_i b_j + \sum_{j=m+1}^{n} a_j b_i, \] respectively, for Cases (a), (b), and (c).

**Problem 2:** For a given \((A, B)\) what is a smallest order (size) solution to Problem 1?

**Theorem 1.4.** If \(A = \{a_1, a_2, \ldots, a_m\}\) and \(B = \{b_1, b_2, \ldots, b_n\}\), then the minimum order \(p_{\text{min}}(A, B)\) of an \((A, B)\)-graph satisfies
\[ n + m \leq \max \{b_n, m\} + \max \{a_m, n\} < p_{\text{min}}(A, B). \]

**Proof.** Since \(a_m\) must be used, \(Y\) must contain at least \(a_m\) vertices. Likewise, since at least one vertex in \(Y\) must have degree \(b_n\), \(X\) must contain at least \(b_n\) vertices. Furthermore, each \(a_i\) in \(A\) must be used at least once and \(X\) must contain at least \(m\) vertices. Similarly, \(Y\) must contain at least \(n\) vertices. Thus, \(|X| \geq \max \{b_n, m\} \geq n\) and \(|Y| \geq \max \{a_m, n\} \geq m\). Therefore,
\[ p_{\text{min}}(A, B) \geq \max \{b_n, m\} + \max \{a_m, n\} \geq n + m. \]

**2. Comments on the General Connected Case**

**Lemma 2.1.** Let \(G_1\) be a connected \((A, B)\)-graph disjoint from \(G_2\) - a connected \((U, V)\)-graph and let \((x, y)\) and \((u, v)\) be edges in \(G_1\) and \(G_2\), respectively. Furthermore, let \(C\) be the following condition:

There exists a pair of edges \((x, y)\) and \((u, v)\) in \(G_1\) and \(G_2\), respectively, such that at least one of \((x, y)\) and \((u, v)\) is not a bridge.

Then,

1. The operation \(\oplus\) that replaces \((x, y)\) and \((u, v)\) with \((x, v)\) and \((u, y)\) yields an \((A \cup U, B \cup V)\)-graph denoted \(G_1 \oplus G_2\), with
\[ V(G_1 \oplus G_2) = V(G_1) \cup V(G_2) \]

and
\[ E(G_1 \oplus G_2) = \left( E(G_1) - (x, y) \right) \cup \left( E(G_2) - (u, v) \right) \cup \{(x, v), (u, y)\}. \]

2. \(G_1 \oplus G_2\) is connected if and only if Condition \(C\) is satisfied and
\(G_1 \oplus G_2\) is not connected if and only if \(G_1\) and \(G_2\) are both trees.

**Proof.** (1) Clearly, the operation \(\oplus\) is well defined and this results in a graph. To show that the result \(G_1 \oplus G_2\) is a \((A \cup U, B \cup V)\)-graph note that \(\oplus\) is vertex degree preserving (and order preserving).

(2) If Condition \(C\) is satisfied and both \((x, y)\) and \((u, v)\) are not bridges, then clearly \(G_1 \oplus G_2\) is connected. Without loss of generality, assume \((x, y)\) is a bridge and \((u, v)\) is not a bridge, then \(G_1 - (x, y)\) consists of two components \(X\) (containing \(x\)) and \(Y\) (containing \(y\)) and the graph \(G_2 - (u, v)\) consists of one component \(Z\) (containing both \(u\) and \(v\)). It follows that any vertex in \(X\) is path-wise connected to \(x, x\) is adjacent to \(v, v\) is path-wise connected to \(u\) (in fact

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connected to any vertex in $Z$), $u$ is adjacent to $y$, and $y$ is path-wise connected to any vertex in $Y$. Thus, $G_1 \oplus G_2$ is connected.

If Condition $C$ is not satisfied, every edge of $G_1$ and $G_2$ is a bridge. A tree is the only connected graph for which every edge is a bridge. Thus, both $G_1$ and $G_2$ are trees. Applying the $\oplus$ operation to two trees for any pair of edges $(x, y)$ and $(u, v)$ will result in a disconnected graph. Therefore, in this case, $G_1 \oplus G_2$ can never be connected. Consequently, $G_1 \oplus G_2$ is connected if and only if Condition $C$ is satisfied. The second part of (2), follows from Condition $C$ is not satisfied is equivalent to both $G_1$ and $G_2$ are trees.

Theorem 2.2. Problem 1 has a connected solution for all pairs $(A, B)$, provided $1 \not\in A$ and $1 \not\in B$, that is, $1 \not\in A \cap B$.

Proof. From Theorem 1.2, given any $A$ and $B$ there exists an $(A, B)$-graph. One such example is the graph $G = \cup H_{i,j}$ constructed as in the proof of Theorem 1.2. Since each $H_{i,j} = K_{b_j,a_i}$ and since neither $b_j$ nor $a_i$ is equal to 1, each $H_{i,j}$ has a non-bridge edge. Applying Lemma 2.1 to any pair of components of $G$ will produce an $(A, B)$-graph $G\# \text{with one component less than } G$. Successively, applying Lemma 2.1 in this manner will lead to a connected $(A, B)$-graph.

Theorem 2.3. Let $A = \{1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. Then, there exists a connected $(A, B)$-graph except when $A = \{1\}$ and $n > 2$ or $B = \{1\}$ and $m > 2$.

Proof. Let $A = \{1\}$, that is, $m \geq 2$ and $b_1 = 1$.
Let $G_1$ be the $(\{1\}, \{b_1\})$-graph realized by a $b_1$-star. Next, starting with $\cup_{2 \leq i \leq m} H_{i,j}$ and applying Lemma 2.1 arrive at a connected $(\{a_2, \ldots, a_m\}, B)$-graph $G_2$. Since $G_2$ contains at least one non-bridge edge, also by Lemma 2.1, $G_1 \oplus G_2$ is a connected $(A, B)$-graph.

Let $A = \{1\}$ and $b_1 = 1$ with $n \geq 2$.
Let $G_1 = K_2$. Starting with $\cup_{2 \leq i \leq m} H_{i,j}$ and applying Lemma 2.1 arrive at a connected $(\{a_2, \ldots, a_m\}, \{b_2, \ldots, b_n\})$-graph $G_2$. Since $G_2$ contains at least one non-bridge edge, $G_1 \oplus G_2$ is a connected $(A, B)$-graph.

The remaining cases are $A = \{1\} \not\in B = \{1\} (m \geq 2)$, $A = \{1\} \not\in B = \{b_1, b_2, \ldots, b_n\}$ $(n \geq 2)$, and $A = \{1\} \not\in B = \{1\}$. In the first two cases, an $(A, B)$-graph is the union of $a$-stars and $b$-stars, respectively, and is not connected. In the third case, a $K_2$ is a connected $(A, B)$-graph realization.

3. Minimum Order Connected $(A, B)$-Graphs
First note the following inequality that was obtained in Theorem 1.4.

$$n + m \leq \max\{b_n, m\} + \max\{a_m, n\} \leq \rho_{\min}(A, B).$$
The smallest order possible for a realizable graph occurs when exactly one element of A and exactly one element of B occurs as the degree of a vertex exactly once. In this case, such a graph, if it exists, will have order $|X| + |Y| = |A| + |B| = m + n$ and size $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = s = t$.

Suppose now that $a_m \geq n$. Then, the vertex of degree $a_m$ in X must join $a_m$ vertices in Y. Since $a_m \geq n$, it may be possible to distribute the $n$ elements of B among the $a_m$ vertices and therefore no additional vertices will be required in Y. In this case, $|Y| = a_m$. Thus, the smallest order possible here is $|X| + |Y| = |A| + a_m = m + a_m$ with corresponding size $\sum_{i=1}^{m} a_i = s$. An analogous argument beginning with B will yield the necessary condition $b_n \geq m$ and a smallest order $n + b_n$ with corresponding size $\sum_{j=1}^{n} b_j = t$. The above discussion leads to the following theorem.

**Theorem 3.1.** Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. If there exists an $(A, B)$-graph $G$ with $p$ vertices and $q$ edges such that

(i) $p = m + n$ and $q = s = t$, or
(ii) $p = m + a_m$ and $q = s$ for $a_m \geq n$ and $b_n \leq m$, or
(iii) $p = n + b_n$ and $q = t$ for $b_n \geq m$ and $a_m \leq n$,

then $G$ will be a smallest order realizable $(A, B)$-graph. ■

Consider, $a_m \geq n$ and $b_n \geq m$. If both inequalities are equalities, then we get $n + m$ as the minimum order. Thus, consider $a_m > n$ and $b_n > m$. Here the minimum order $p = a_m + b_n$ follows from $\max\{b_n, m\} + \max\{a_m, n\}$. However, the size $q$ cannot be specified, other than $q > \max\{s, t\}$.

Observe that $a_m \geq m$ and $b_n \geq n$. Thus, cannot have both $m > b_n$ and $n > a_m$, since this leads to the contradiction $a_m \geq m > b_n \geq n > a_m$. Similarly, starting out with $b_n$ leads to $b_n > b_n$. Note that this does not mean $n + m$ cannot be realized as a minimum order for some $(A, B)$-graph (see Example 3.1).

The following examples show that there are minimum order connected realizations that correspond to $n + m$, $b_n + n$, $m + a_m$, and $b_n + a_m$.

**Example 3.1.** Let $A = \{1, 2\}$ and $B = \{1, 2\}$. Then, see Figure 3.1,

$\rho_{\min}(A, B) = n + m = b_n + a_m = b_n + n = m + a_m$.

This generalizes to $A = \{1, 2, 3, \ldots, m\}$ and $B = \{1, 2, 3, \ldots, n\}$ when $m = n$. 
Example 3.2. Let $A = \{1, 3\}$ and $B = \{3, 4\}$. Then
\[ p_{\text{min}}(A, B) = a_m + b_n > n + b_n > m + a_m > m + n. \]
See Figure 3.2.

Example 3.3. Let $A = \{1, 2, 4\}$ and $B = \{1, 2\}$. Then
\[ p_{\text{min}}(A, B) = m + a_m > b_n + a_m > m + n > b_n + n. \]
See Figure 3.3.

Example 3.4. Let $A = \{2\}$ and $B = \{1, 2, 3\}$. Then
\[ p_{\text{min}}(A, B) = b_n + n > b_n + a_m > m + n > m + a_m. \]
See Figure 3.4.

Figure 3.1. A minimum order (\{1, 2\}, \{1, 2\})-graph

Figure 3.2. A minimum order (\{1, 3\}, \{3, 4\})-graph

Figure 3.3. A minimum order (\{1, 2, 4\}, \{1, 2\})-graph
It is further noted that that the bound
\[ \max\{b_n, m\} + \max\{a_m, n\} \leq p_{\text{mid}}(A, B) \]
given in Theorem 1.4 is not necessarily an equality. This is seen by considering
\( A = \{2, 3, 4\} \) and \( B = \{1, 2\} \). Here, it is easy to verify that a minimum order
realization of an \((A, B)\)-graph requires 8 vertices (see Figure 3.5). Thus,
\[ 7 = \max\{2, 3\} + \max\{4, 2\} < p_{\text{mid}}(A, B) = 8. \]

Figure 3.5. A minimum order \((\{2, 3, 4\}, \{1, 2\})\)-graph

4. \((A, B)\)-Graphs When \(B\) is a Singleton

**Theorem 4.1.** Let \( p_{\text{mid}}(A, B) \) denote the minimum order of an \((A, B)\)-graph with
\( A = \{a_1, a_2, \ldots, a_m\}, \ B = \{b\}, \) and \( b \neq 1 \). Then,
\[ \max\{b, m\} + a_m \leq p_{\text{mid}}(A, B) \leq s + mb. \]

**Proof.** The construction of the graph \( G^* = \cup H_{i,j} \) in the proof of Theorem 1.3
provides an upper bound for the minimum order of an \((A, B)\)-graph in this case,
\[ \sum_{j=1}^{m} a_j + b + (m - 1)b = s + mb. \] A lower bound in this case is obtained by applying
Theorem 1.4, so that
\[ \max\{b, m\} + \max\{a_m, n\} = \max\{b, m\} + a_m \leq p_{\text{mid}}(A, B). \]

**Theorem 4.2.** If \( A = \{a_1, a_2, \ldots, a_m\}, \ B = \{1\}, \) then an \((A, B)\)-graph is a union of
stars.

**Corollary 4.3.** If \( A = \{a\} \) and \( B = \{1\}, \) then the unique minimum order (size)
\((A, B)\)-graph is the star \( S(t(a)). \)

**Theorem 4.4.** If \( A = \{a\} \) and \( B = \{b\}, \) then the unique minimum order (size)
\((A, B)\)-graph is the complete bipartite graph \( K_{b,a} \) with order \( b + a \) and size \( ab. \)
5. A Linear Programming Approach to the Minimum Order Problem

Let X and Y denote the bipartition sets of an (A, B)-graph where A = \{a_1, a_2, \ldots, a_m\} and B = \{b_1, b_2, \ldots, b_n\}, then the following conditions lead to an integer linear programming formulation of the minimum order problem.

Let \(x_i\) denote the number of vertices in X with degree \(a_i\) and \(z_j\) denote the number of vertices in Y with degree \(b_j\). Since the sum of the degrees of the vertices in Y must equal the sum of the degrees of the vertices in X

\[a_1 x_1 + a_2 x_2 + \ldots + a_m x_m = b_1 z_1 + b_2 z_2 + \ldots + b_n z_n.\]

The number of vertices in X must be at least equal to \(b_n\); and at least equal to the number \(m\) of elements in A. Thus

\[x_1 + x_2 + \ldots + x_m \geq \max\{b_n, m\}.\]

Similarly, by considering the set Y one obtains

\[z_1 + z_2 + \ldots + z_n \geq \max\{a_m, n\}.\]

Since each element of A must be used at least once and each element of B must be used at least once

\[x_i \geq 1 \quad (1 \leq i \leq m) \quad \text{and} \quad z_j \geq 1 \quad (1 \leq j \leq n).\]

For an (A, B)-graph to have minimum order the number of vertices must be minimum. Thus the following expression must be minimized

\[x_1 + x_2 + \ldots + x_m + z_1 + z_2 + \ldots + z_n.\]

The above observations lead to the following theorem.

**Theorem 5.1.** A linear program for the solution of the minimum order problem for (A, B)-graphs with A = \{a_1, a_2, \ldots, a_m\} and B = \{b_1, b_2, \ldots, b_n\}, is

\[
\begin{align*}
\text{Minimize } & x_1 + x_2 + \ldots + x_m + z_1 + z_2 + \ldots + z_n \\
\text{with } & x_i \ (1 \leq i \leq m) \text{ and } z_j \ (1 \leq j \leq n) \text{ integral} \\
\text{subject to } & x_i \geq 1 \\
& z_j \geq 1 \\
& x_1 + x_2 + \ldots + x_m \geq \max\{b_n, m\} \\
& z_1 + z_2 + \ldots + z_n \geq \max\{a_m, n\} \\
& a_1 x_1 + a_2 x_2 + \ldots + a_m x_m = b_1 z_1 + b_2 z_2 + \ldots + b_n z_n.
\end{align*}
\]
Corollary 5.2. A linear program for the solution of the minimum order problem for \((A, B)\)-graphs with \(A = \{a_1, a_2, \ldots, a_m\}\) and \(B = \{b\}\) is

Minimize \(x_1 + x_2 + \ldots + x_m + z\)
with \(x_i\) \((1 \leq i \leq m)\) and \(z\) integral
subject to
\[x_i \geq 1\]
\[x_1 + x_2 + \ldots + x_m \geq \max\{b, m\}\]
\[z \geq a_m\]
\[a_1 x_1 + a_2 x_2 + \ldots + a_m x_m = bz.\]

Proof. Special case of Theorem 5.1. \(\blacksquare\)

Corollary 5.3. A linear program for the solution of the minimum order problem for \((A, B)\)-graphs with \(A = \{a_1, a_2\}\) and \(B = \{b\}\) is

Minimize \(x_1 + x_2 + z\)
with \(x_1, x_2,\) and \(z\) integral
subject to
\[x_i \geq 1\quad i = 1, 2\]
\[x_1 + x_2 \geq \max\{b, 2\}\]
\[z \geq a_2\]
\[a_1 x_1 + a_2 x_2 = bz.\]

Proof. Special case of Theorem 5.1. \(\blacksquare\)

Example 5.1. Let \(A = \{1, 3\}\) and \(B = \{2\}\). Apply Corollary 5.3 to obtain solution.

Minimize \(x_1 + x_2 + z\)
with \(x_1, x_2,\) and \(z\) integral
subject to
\[x_1 \geq 1\]
\[x_2 \geq 1\]
\[x_1 + x_2 \geq 2\) (redundant)
\[z \geq 3\]
\[x_1 + 3x_2 = 2z.\]

Solution by inspection: Since the objective function is just the sum of the three variables \(x_1, x_2,\) and \(z,\) constraints 1, 2, and 4 show that a lower bound for the
objective function is 5. However, \((x_1, x_2, z) = (1, 1, 3)\), does not satisfy constraint 5, thus is not feasible. Incrementally increasing the sum \(x_1 + x_2 + z\) one can find the first occurrence of a feasible triple. If each of the variables is individually raised by 1, namely, \((2, 1, 3), (1, 2, 3), (1, 1, 4)\), again constraint 5 is not satisfied. If the sum is raised by 2, that is, \((2, 2, 3), (2, 1, 4), (1, 2, 4), (3, 1, 3), (1, 3, 3), (1, 1, 5)\), it is seen that \((3, 1, 3)\) is the only triple that also satisfies constraint 5. Thus, \((3, 1, 3)\) is the unique minimum solution yielding a value of 7. The \((A, B)\)-graph realizing this extrema is shown in Figure 5.1.

![Figure 5.1.](image)

**Remark.** Solving the linear program related to a given pair \((A, B)\) will yield a degree distribution \(x_1, x_2, \ldots, x_m\) corresponding to one of the partite vertex sets and a degree distribution \(z_1, z_2, \ldots, z_n\) corresponding to the other partite vertex set. This degree distribution will correspond to a minimum order \((A, B)\)-graph. However, the construction of such a graph is not provided. Nevertheless, the linear programming solution is a good start for obtaining a solution to the minimum order problem. On the other hand, if an \((A, B)\)-graph is presented as a candidate solution to the minimum order problem; the linear program solution provides a confirmation of the solution.

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