NEIGHBORHOOD REGULAR GRAPHS

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Abstract. A regular graph G is called *vertex transitive* if the automorphism group of G contains a single orbit. In this paper we define and consider another class of regular graphs called *neighborhood regular graphs* abbreviated NR. In particular, let G be a graph and $N[v]$ be the closed neighborhood of a vertex v of G. Denote by $G(N[v])$ the subgraph of G induced by $N[v]$. We call G NR if $G(N[v]) \cong G(N[v'])$ for each pair of vertices v and v' in $V(G)$. A vertex transitive graph is necessarily NR. The converse, however, is in general not true as is shown by the union of the cycles $C_4 \cup C_5$. Here we provide a method for constructing an infinite class of connected NR graphs which are not vertex transitive. A NR graph G is called *neighborhood regular relative to* N if $N[v] \cong N$ for each $v \in V(G)$. Necessary conditions for N are given along with several theorems which address the problem of finding the smallest order (size) graph that is NR relative to a given N. A table of solutions to this problem is given for all graphs N up to order five.

1. Introduction

A graph G is called *vertex transitive* if the automorphism group of G, $\mathrm{Aut}(G)$, contains a single orbit. Examples of vertex transitive graphs are the n-cycle, C_n, and the complete graph, K_n, on n vertices. Observe that these graphs are also regular: The degree of each vertex of C_n is two, while the complete graph, K_n, is regular of degree $n-1$. This is generally the case as *every vertex transitive graph is regular.* To show this, let v and v' be distinct vertices of a vertex transitive graph G, and $\phi \in \mathrm{Aut}(G)$ an automorphism of G such that $\phi(v) = v'$. The result follows from the fact that ϕ preserves adjacencies in G. Note that the set of vertex transitive graphs is properly contained in the set of regular graphs as not every regular graph is vertex transitive. The graph of Figure 1.1 is regular but not vertex transitive as no automorphism of G maps the vertex u into v.

In this paper we define and consider another class of regular graphs called *neighborhood regular graphs*. As a preliminary step we review some terminology. Let G be a graph and v a vertex of G. A *neighbor* of v is a vertex that is adjacent to v in G. The *open neighborhood* of v, denoted by $N(v)$, is the set of neighbors of v. That is, $N(v) = \{u \in V(G) \mid \{u, v\} \in E(G)\}$. The *closed neighborhood* of v, denoted by $N[v]$, is defined by $N[v] = N(v) \cup \{v\}$. The subgraphs of G induced by the open and closed neighborhoods of v, which we refer to as the as the *open* and *closed neighborhood subgraphs of* v, are denoted, respectively, by $G(N(v))$ and $G(N[v])$.

Definition 1.1. A graph G is called *neighborhood regular*, abbreviated NR, if the closed neighborhood graphs of each vertex of G are isomorphically the same. That is, G is NR if and only if $G(N[v]) \cong G(N[v'])$ for each pair of vertices v and v' in $V(G)$.

We say that a graph G is *neighborhood regular relative to* N if the closed neighborhood graph of each vertex of G is isomorphic to N. For examples of NR graphs consider again the cycle C_n and the complete graph K_n on n vertices. In particular, observe that C_3 is neighborhood regular relative to itself. For $n \geq 4$, we have that the closed neighborhood graph for each vertex of C_n is P_3, the path of order 3, so that for $n \geq 4$, C_n is neighborhood regular relative to P_3. The closed neighborhood graph for each vertex of K_n is K_n so that K_n is NR relative to itself.

Obviously, every NR graph is also regular, since if G is NR relative to N then $|N[v]| = |N|$ for each vertex v in G. However, the converse of this statement does not hold. To show this we again consider the graph of Figure 1.1. Observe that in this graph $G(N[u])$ is $K_{1,3}$ whereas $G(N[v])$ is K_3 with a pendant vertex.

The next result follows from the fact that the vertices of a vertex transitive graph are indistinguishable.

Theorem 1.1. *Every vertex transitive graph is also neighborhood regular.*

Proof: Let v and v' be arbitrary vertices of G, and ϕ an automorphism of G such that $\phi(v) = v'$. Since ϕ preserves the adjacencies of v and its neighbors then $\phi(N[v]) = N[\phi(v)] = N[v']$. So that the closed neighborhood subgraph of v is mapped to the closed neighborhood subgraph of v'. To complete the proof let x and y be vertices in the open neighborhood

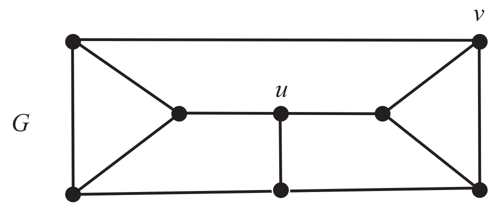


FIGURE 1.1. A regular graph that is not vertex transitive nor neighborhood regular

subgraph of v. Since ϕ also preserves adjacencies in $N(v)$ then x is adjacent to y in $N(v)$ if and only if $\phi(x)$ is adjacent to $\phi(y)$ in $N(v')$. Hence $G(N[v]) \cong G(N[v'])$ as desired. ■

Next we show that the converse of Theorem 1.1 does not hold in general and that the set of vertex transitive graphs is properly contained in the set of NR graphs. First, observe that the graph G of Figure 1.2 is NR relative to $K_{1,3}$. However, G is not vertex transitive as no automorphism maps the vertex w to v since w belongs to a square (actually several) while v belongs to no square of G.

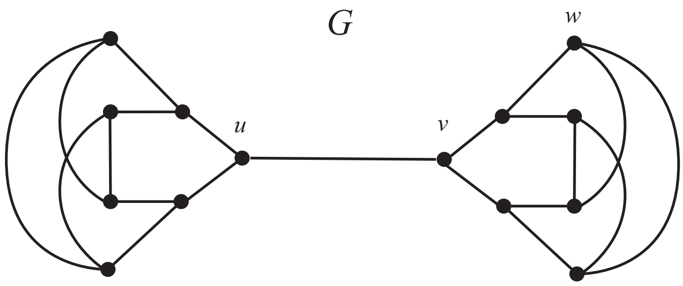


FIGURE 1.2. NR graph that is not vertex transitive

The graph G of Figure 1.2 can be augmented to produce an infinite class of NR graphs that are not vertex transitive. To begin the construction observe that the removal of the bridge $\{u, v\}$ from G separates G into two identical connected components H and H'. Let H'' be a third copy. Augment G by adding a vertex x and replacing edge $\{u, v\}$ with $\{u, x\}$ and $\{x, v\}$. Then connect x by an edge to the vertex y of degree two in H''. Figure 1.3 gives a depiction of the result.

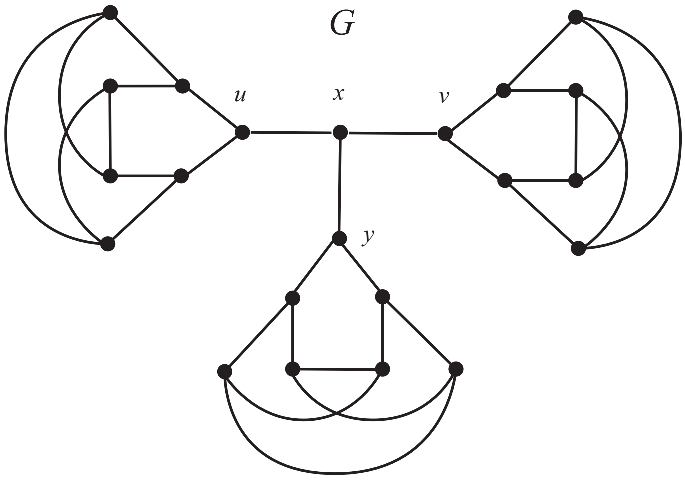


Figure 1.3. A NR graph G not vertex transitive with $|G| = 22$.

If we repeat this procedure on any of the edges $\{u, x\}$, $\{x, v\}$, or $\{x, y\}$ we obtain a similar result for $|G| = 30$. Continuing in this way we obtain a class of NR graphs that are not vertex transitive with $|G| = 14 + 8n$ for $n \in \{1, 2, 3, \ldots\}$.

Another construction, involving the union of cycles, gives an infinite class of NR graphs that are not vertex transitive. First, observe that $G = C_4 \cup C_5$ is NR (relative to P_3) but is not vertex transitive since no automorphism maps a vertex from C_4 to a vertex of C_5. The next proposition generalizes this idea.

Theorem 1.2. *Let $n \geq 9$. The graphs G defined by*

$$\bigcup_{j=1}^{k} C_{n_j}$$

with $k \geq 2, n_j \geq 4$, and not any n_j equal are all NR (relative to P_3) but are not vertex transitive.

The following figure summarizes the above results.

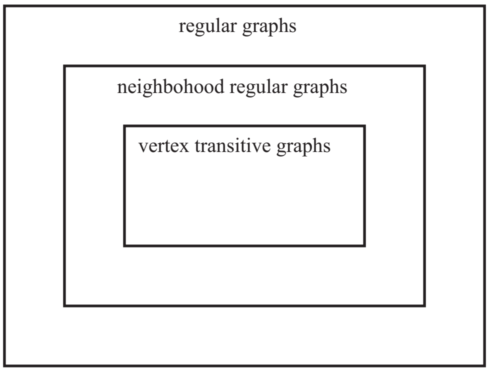


FIGURE 1.4. Containment properties of regular graphs, NR graphs, and vertex transitive graphs.

2. Open Problems and Partial Results

In this section we consider several problems regarding NR graphs. The first open problem is to find an infinite class of connected NR graphs of arbitrary order n. A first step towards the solution was taken with the construction of Figure 1.3. We now articulate several other problems.

Problem 1: For each r, find the smallest order (size) connected r-regular graph that is not NR. Observe that the graph of G of Figure 1.1 (a candidate for the $r = 3$ solution) is such that $|G| = 8$ and $|\mathrm{E}(G)| = 12$.

Problem 2: For each r, find the smallest order (size) connected NR graph that is not vertex transitive. The graph of Figure 1.2 is a candidate for the $r = 3$ solution of this problem.

Problem 3: Given N, what is the smallest order (size) connected graph G that is NR relative to N.

We proceed with some partial results related to Problem 3. Before doing so, we note that to be considered, N must satisfy certain conditions.

(1) N must be connected
(2) N must contain a spanning $(|N| - 1)$-star.
(3) All pendant vertices in N must be incident to the same vertex and this vertex has degree $|N| - 1$.

The next theorem follows immediately from the second condition.

Theorem 2.1. *There is no graph G which is NR relative to P_n or C_n if $n > 3$. For $n \leq 3$ observe that P_1, P_2 and C_3 are all NR relative to themselves, while C_4 is NR relative to P_3.*

The condition that N contain a spanning star can also be used to show that there is no graph which is NR relative to any tree T with $\text{diam}(T) \geq 3$. We have already mentioned that K_n is NR relative to itself. Thus K_n is a solution to Problem 3 for $N = K_n$.

In the next theorem we consider Problem 3 when N is itself a star. We require the following theorem due to Turan.

Theorem 2.2. [2] *The maximum number of edges among all graphs of order n with no triangles is $\frac{n^2}{4}$.*

Theorem 2.3. *The complete bipartite graph, $K_{r,r}$, is a solution to Problem 3 if N is the star $St(r)$.*

Proof. Let G be a solution to Problem 3 for $St(r)$ and let n be the order of G. Since G is r-regular then $rn = 2e$ where e is the size of G. Observe that G must be free of triangles. Applying Lemma 1 we have that

$$\frac{kn}{2} \leq \frac{n^2}{4}$$

or $n \geq 2r$. This implies that $e \geq r^2$. To complete the proof observe that the order and size of $K_{r,r}$ is $2r$ and r^2, respectively, thereby achieving the minimum value for both of these parameters. Note also that as $K_{r,r}$ is bipartite it is free of any odd cycles. In particular, it contains no triangles. □

The next class of graphs that we consider for use as N are triangles with k pendant vertices adjacent to a single vertex of the triangle. See Figure 2.1.

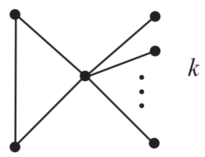


FIGURE 2.1. Triangle with k pendant vertices.

For $k = 1$ we have that $G = K_3 \times K_2$ is NR relative to N when it is a triangle with a single pendant vertex. Moreover, G is the smallest graph with this property and is thus a solution to Problem 3 for this choice of N.

We also have that the graph $G = K_3 \times K_{2,2}$ is a solution to Problem 3 when N is a triangle with two pendant vertices incident with a single vertex of the triangle, (see Table 1 for a depiction of G). The next theorem is a first step towards a generalization.

Theorem 2.4. *Let N_k be a triangle with k pendant vertices adjacent to a single vertex. Then $G = K_3 \times K_{k,k}$, is NR relative to N_k.*

Proof. Let v be a vertex of G. First observe that v belongs to exactly one of the three copies of $K_{k,k}$ contained in G. Denote this copy by $K^v_{k,k}$. First, if v is an element of say the *white* color set of $K^v_{k,k}$, then v is adjacent to each vertex of the *black* color set of $K^v(k,k)$. We also have that v is contained in a triangle. Affix the labels x and y to the other two vertices of this triangle. As neither x nor y is adjacent to any of the "black" vertices of $K^v_{k,k}$ then the neighborhood graph of v is N_k. To complete the proof observe that G is vertex transitive as it is the Cartesian product of two vertex transitive graphs. □

The same method of proof yields the following theorem.

Theorem 2.5. *Let $N_{n,k}$ be a K_n with k pendant vertices adjacent to a single vertex. Then $G = K_n \times K_{k,k}$, is NR relative to $N_{n,k}$.*

For an example we have that $G = K_4 \times K_2$ is NR relative to N=K_4 with a single pendant vertex. As G is the smallest graph with this property it is a solution to Problem 3 for this choice of N, (see Table 1.)

3. Solutions to Problem 3 for Small Graphs

The following table gives illustrations of the solutions to Problem 3 for all connected graphs up to order 5.

Table 3.1: Solutions to Problem 3

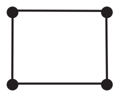
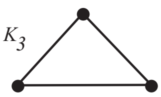
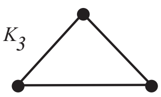
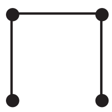
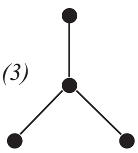
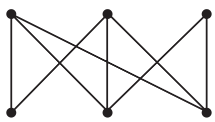
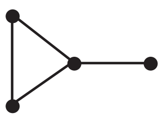
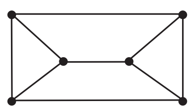
$\|N\|$	N	G	$\|G\|$
1	K_1	K_1	1
2	K_2 	K_2 	2
3	P_3 	C_4 	4
	K_3 	K_3 	–
4	P_4 	Does not exist	4
	$St(3)$ 	$K_{3,3}$ 	6
		$K_3 \times K_3$ 	6

Table 3.1 Solutions to Problem 3 (Continued)

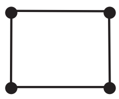
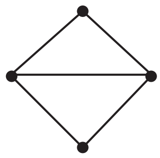
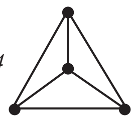
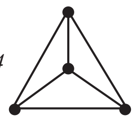
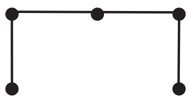
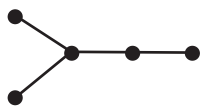
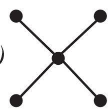
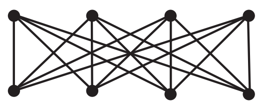
$\lvert N \rvert$	N	G	$\lvert G \rvert$
4 (cont.)	C_4 	Does not exist	–
		Does not exist	–
	K_4 	K_4 	4
5	P_5 	Does not exist	–
		Does not exist	–
	$St(4)$ 	$K_{4,4}$ 	8

Table 3.1 Solutions to Problem 3 (Continued)

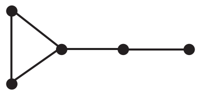
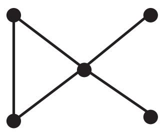
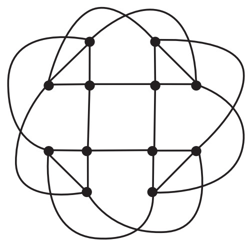
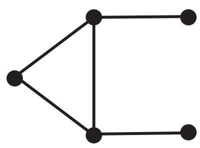
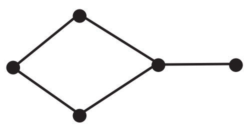
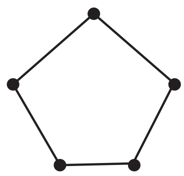
$\lvert N\rvert$	N	G	$\lvert G\rvert$
5 (cont.)		Does not exist	–
		$K_3 \times C_4$ 	12
		Does not exist	–
		Does not exist	–
	C_5 	Does not exist	–

Table 3.1 Solutions to Problem 3 (Continued)

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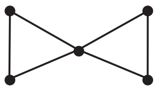
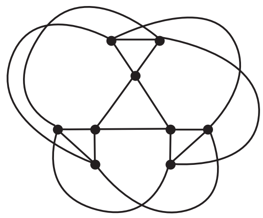
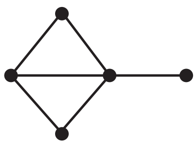
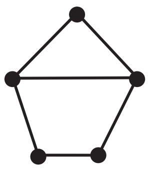
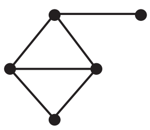
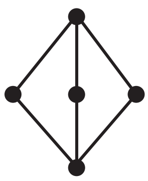
5 (cont.)		$K_3 \times K_3$ 	9
		Does not exist	–
		Does not exist	–
		Does not exist	–
		Does not exist	–

Table 3.1 Solutions to Problem 3 (Continued)

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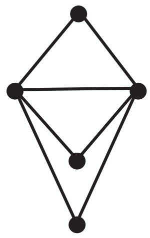
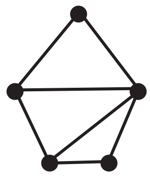
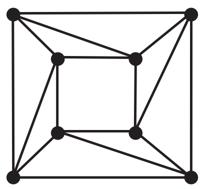
5		Does not exist	–
			8

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[2] P. Turan. Eine extremalaufgabe aus der graphentheorie. *Mat. Fiz. Lapok*, 48:436–452, 1941.