Escher was quite excited by that figure, since it showed him how to solve a problem that he had wondered about for a long time: how to create a repeating pattern within a limiting circle, so that the basic subpatterns or motifs became smaller toward the circular boundary.

A Tale Both Shocking and Hyperbolic

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Dutch artist M. C. Escher wrote in 1958, upon seeing the pattern of Figure 1, that it "gave me quite a shock." This pattern of curvilinear triangles appeared in a paper by the Canadian geometer H.S.M. Coxeter entitled *C rystal Symmetry and Its Generalizations*. Coxeter, and most likely other mathematicians before him, drew such patterns by using classical straightedge and compass constructions. Exactly how this was done was a "folk art" until recently when it was explained by Chaim Goodman-Strauss.

To explain how Escher came to be shocked, we go back a few years earlier to the 1954 International Congress of Mathematicians, where Coxeter and Escher first met. This led to friendship and correspondence. A couple of years after their first meeting, Coxeter wrote Escher asking for permission to use some of his striking designs in a paper on symmetry, *Crystal Symmetry and Its Generalizations* (published in the *Transactions of the Royal Society of Canada* in 1957). As a courtesy, Coxeter sent Escher a copy of that paper containing a figure with a hyperbolic tessellation just like that in Figure 1 (in addition to Escher's designs). Escher was quite excited by that figure, since it showed him how to solve a problem that he had wondered about for a long time: how to create a repeating pattern within a limiting circle, so that the basic subpatterns or *motifs* became smaller toward the circular boundary. Escher wrote back to Coxeter telling of his "shock" upon seeing the figure, since it showed him at a glance the solution to his long-standing problem.

Escher, no stranger to straightedge and compass constructions, was able to reconstruct the circular arcs in Coxeter's figure. He put these constructions to good use in creating his first circle limit pattern, *Circle Limit I* which he included with his letter to Coxeter. Figure 2 shows a rough computer rendition of that pattern.

It is easy to see that Figures 1 and 2 are related. Here is how Escher might have created *C i rcle Limit I* from Figure 1. First switch the colors of half the triangles of Figure 1 so that triangles sharing a hypotenuse have the same color. The result is a pattern of "kites," as shown in Figure 3.

Next, remove small triangular pieces from each of the short sides of the orange kites and paste them onto the long sides. Figure 4 shows the result of doing this for just the central kites. This produces the outlines of the blue fish in *Circle Limit I*. The outlines of the white fish are formed by the holes between the blue fish.

Finally, the pattern of *C i rcle Limit I* can be reconstructed by filling in the interior details such as the eyes and backbones.

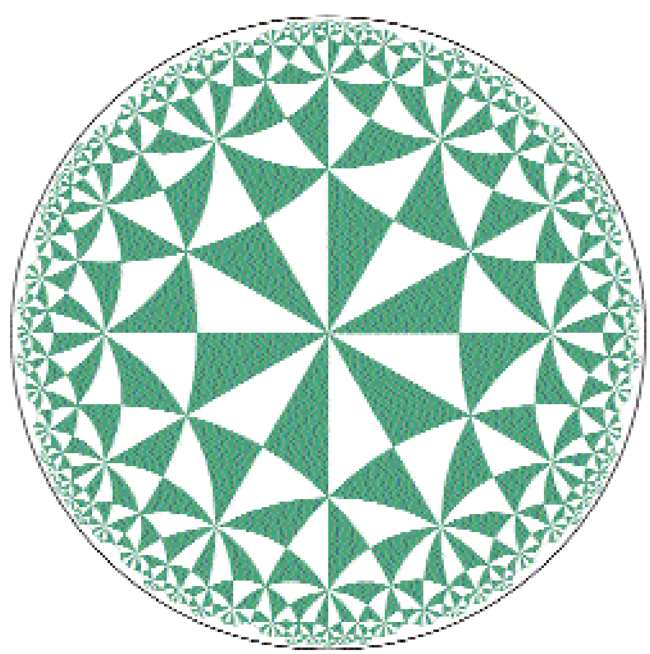


Figure 1. A tessellation of the hyperbolic plane by 30-45-90 triangles.

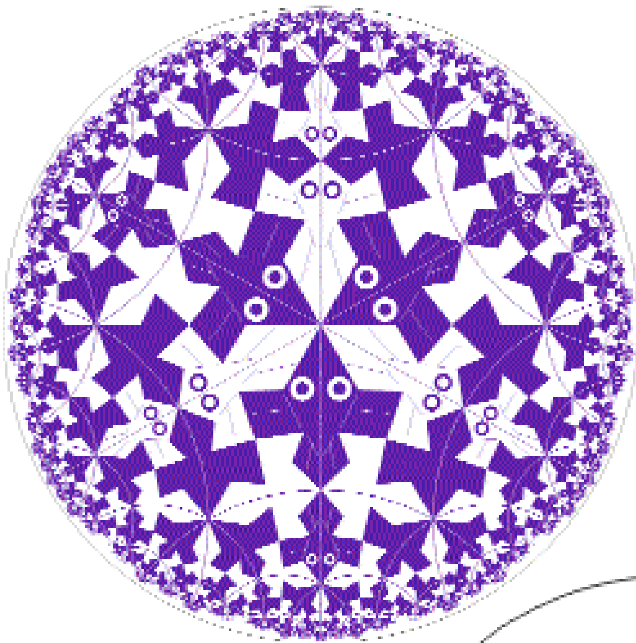


Figure 2. A computer rendition of the pattern in Escher's print *Circle Limit I*.

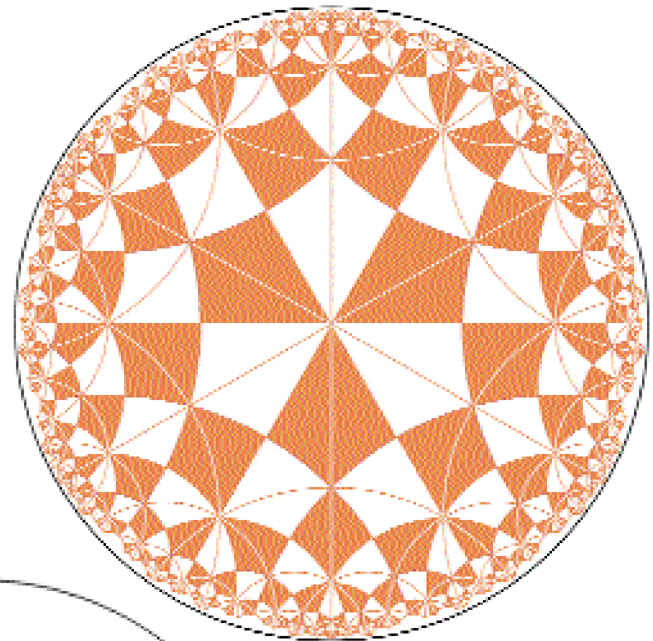


Figure 3. A pattern of "kites" derived from Figure 1.

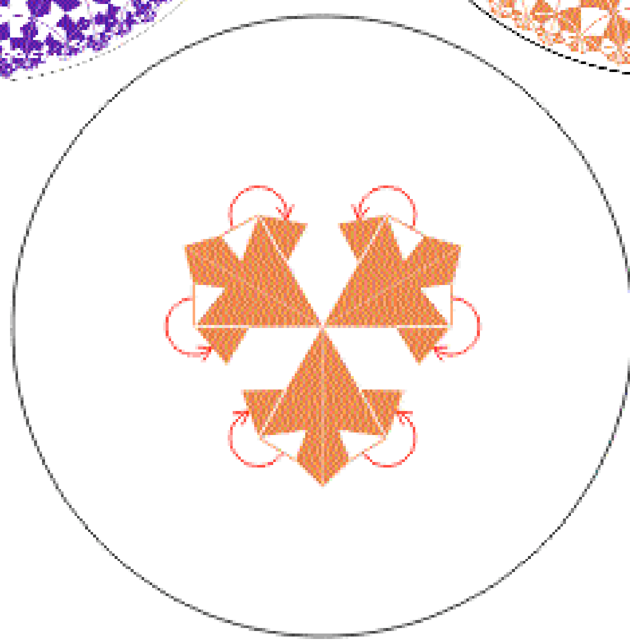


Figure 4. The outlines of the central fish formed from the kites of Figure 3.

A Bit of Hyperbolic Geometry

Mathematicians, and geometers in particular, will recognize the patterns of the figures above as "living in" the Poincaré disk model of hyperbolic geometry. Escher probably knew this, but wasn't concerned about it since he could use Euclidean constructions to build his patterns. The fact that Poincaré's disk model can be defined in purely Euclidean terms shows that hyperbolic geometry is just as consistent as Euclidean geometry. But that is another story.

The *points* of the Poincaré disk model of hyperbolic geometry are the interior points of a bounding circle in the Euclidean plane. In this model, hyperbolic *lines* are represented by circular arcs that are perpendicular to the bounding circle, including diameters. Figures 1 and 2 show examples of these perpendicular circular arcs. Equal hyperbolic distances are represented by ever smaller Euclidean distances as one approaches the bounding circle. For example, all the triangles in Figure 1 are the same hyperbolic size, as are all the blue fish (or white fish) of Figure 2, and the kites of Figure 3. The patterns of Figures 1, 2, and 3 are closely related to the regular hyperbolic tessellation $\{6,4\}$ shown in Figure 5. In general, $\{p,q\}$ denotes the regular tessellation by regular p-sided polygons with q of them meeting at each vertex.

Escher's Criticisms of *Circle Limit I*

Escher had several criticisms of his *Circle Limit I* pattern. First, the fish are "rectilinear," instead of having the curved outlines of real fish. Also, there is no "traffic flow" along the backbone lines - the fish change directions after two fish, and the fish change colors along lines of fish. Another criticism, which Escher didn't make, is the pattern does not have color symmetry since the blue and white fish are not congruent. Before reading further, look back at Figure 2 and try to see why this is true. There are several differences in the shapes of the blue and white fish; the most obvious is the difference in their nose angles.

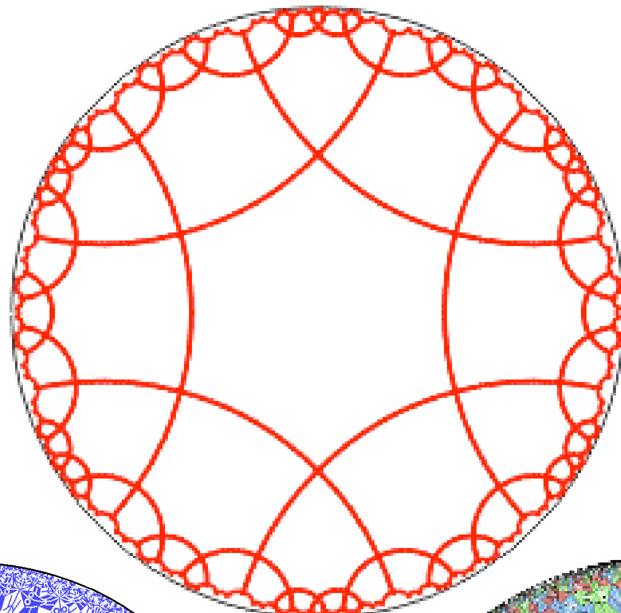
Some of Escher's criticisms could be overcome by basing the fish pattern on the $\{6,6\}$ tessellation, as shown in Figure 6. In fact, Figure 6 can be recolored in three colors to give it color symmetry, which means that every symmetry (rotation, reflection, etc.) of the uncolored pattern exactly permutes the colors of the fish in the colored pattern. Figure 7 shows that three-colored pattern, which addresses all of Escher's criticisms except for the rectilinearity of the fish.

Escher's Solution: *Circle Limit III*

Escher could have used the methods above to overcome his criticisms, but he didn't. Escher took a different route, which led to his beautiful print *Circle Limit III*. Figure 8 shows an approximate computer-generated version of the *Circle Limit III* pattern.

Escher never publicly explained how he designed *Circle Limit III* but here is how he might have gone about it. From his correspondence with Coxeter, Escher knew that regular hyperbolic tessellations $\{p,q\}$ existed for any p and q satisfying $(p-2)(q-2) > 4$. In particular, he had used the $\{8,3\}$ tessellation as the basis for his second hyperbolic pattern, *Circle Limit II*, and he decided to use that tessellation again for *Circle Limit III*. The $\{8,3\}$ tessellation is shown by the heavy red lines in Figure 9.

Figure 5. The regular tessellation $\{6,4\}$ of the hyperbolic plane.



Here is one way to get from the $\{8,3\}$ tessellation to *Circle Limit III*. First, connect alternate vertices of the octagons with slightly curved arcs, which are shown as black arcs in Figure 9. This divides up the hyperbolic plane into "squares" and "equilateral" triangles.

Then if we orient the arcs by putting arrowheads on one end, we get the paths of the fish in *Circle Limit III*. This is shown in Figure 10.

A number of years ago when I was trying to figure out how to encode the color symmetry of *Circle Limit III* in my computer program, I drew a pattern of colored arrows as in Figure 10. Later, in 1998, it was my turn to be "shocked" at the Centennial Exhibition of Escher's works when I saw a colored sketch of arrows by Escher just like mine! He had used his drawing in preparation for *Circle Limit III*.

A Bit More Hyperbolic Geometry

It is tempting to guess that the white backbone lines in Figure 8 are hyperbolic lines (i.e. circular arcs perpendicular to the bounding circle). But careful measurements of *Cir*-

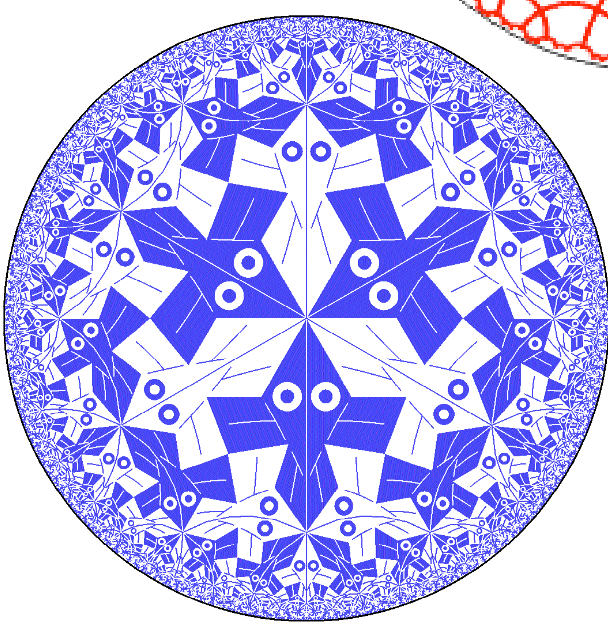


Figure 6. A pattern of rectilinear fish based on the $\{6,6\}$ tessellation.

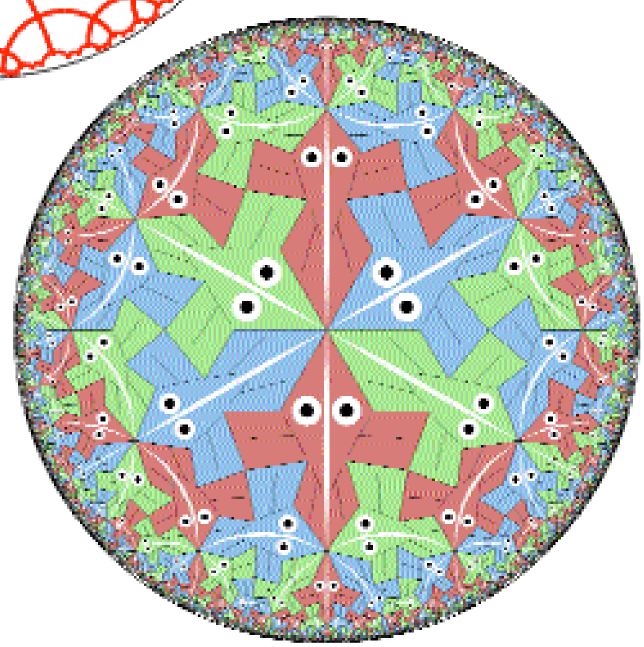


Figure 7. A pattern of rectilinear fish with 3-color symmetry.

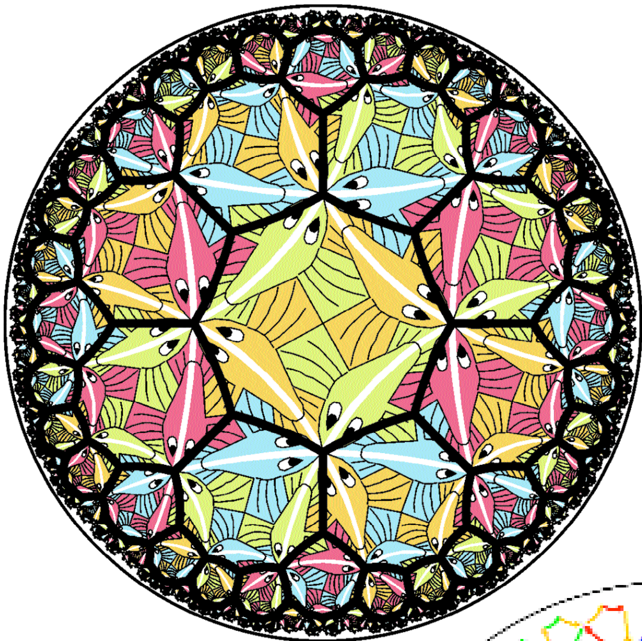


Figure 8. A computer rendition of the *Circle Limit III* pattern and the {8, 3} tessellation (black arcs) upon which it is based.

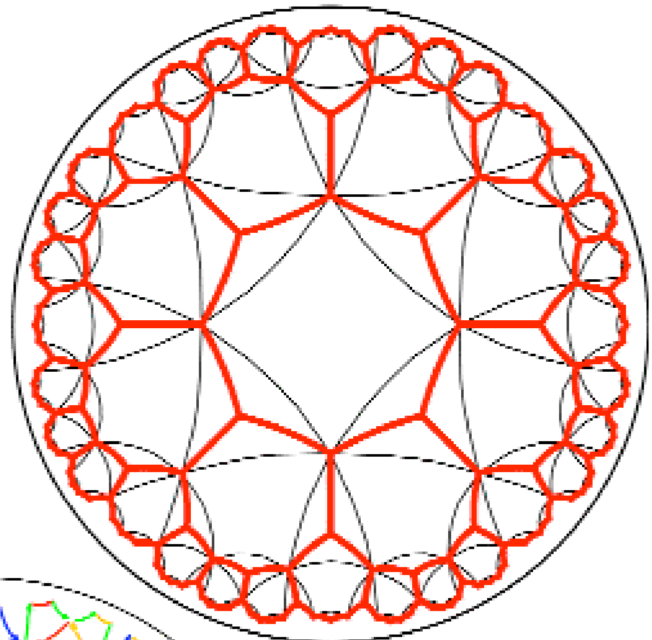


Figure 9. The {8, 3} tessellation (heavy red lines) together with "squares" and triangles (black lines).

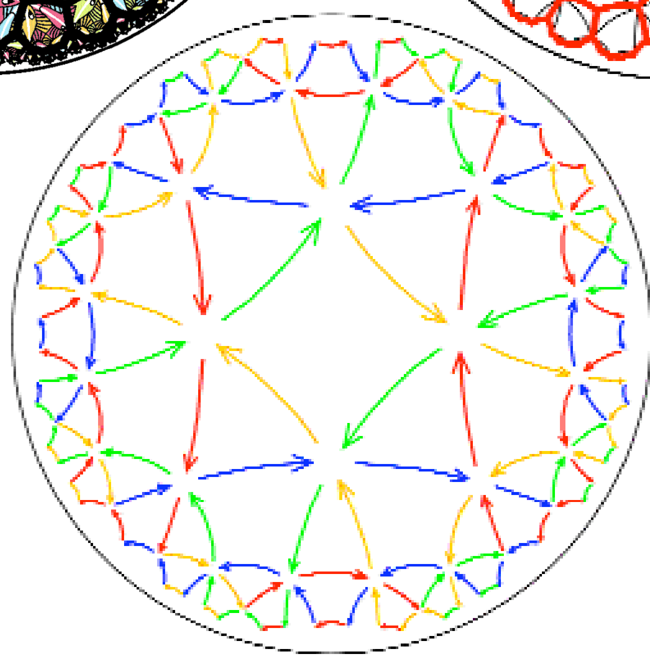


Figure 10. The paths of the fish in *Circle Limit III*.

cle Limit III show that all the white arcs make angles of approximately 80 degrees with the bounding circle. This is as it should be, since the backbone arcs are not hyperbolic lines, but *equidistant curves*, each point of which is an equal hyperbolic distance from a hyperbolic line.

In the Poincaré model, equidistant curves are represented by circular arcs that intersect the bounding circle in acute (or obtuse) angles. Points on such arcs are an equal hyperbolic distance from the hyperbolic line with the same endpoints on the bounding circle. For any acute angle and hyperbolic line, there are two equidistant curves ("branches"), one on each side of the line, making that angle with the bounding circle. Equidistant curves are the hyperbolic analog of small circles in spherical geometry. For example, every point on a small circle of latitude is an equal distance from the equatorial great circle; and there is another small circle in the opposite hemisphere the same distance from the equator.

Each of the backbone arcs in *Circle Limit III* makes the same angle Ω with the bounding circle. Coxeter used hyperbolic trigonometry to show that Ω is given by the following expression:

$$\cos \Omega = \sqrt{\frac{3\sqrt{2}-4}{8}}$$

The value of Ω is about 79.97 degrees, which Escher accurately constructed to high precision.

The 2003 MAM Poster Pattern

Much as Escher was inspired by Coxeter's figure, I was inspired by Escher's "Circle Limit" patterns to create a program that could draw them. More than 20 years ago two students, David Witte and John Lindgren, and I succeeded in writing such a program. Having gone to all the trouble to design a program that was more general than we needed to accomplish our goal, we put it to other uses. *Circle Limit III* is certainly Escher's most stunning hyperbolic pattern, so we thought it would be interesting to find related patterns.

Here is my analysis of *Circle Limit III* fish patterns: one can imagine a three parameter family (k, l, m) in which k right fins, l left fins, and m noses meet, where m must be odd so that the fish swim head to tail. The pattern would be hyperbolic, Euclidean, or spherical depending on whether $1/k + 1/l + 1/m$ is less than, equal to, or greater than 1. *Circle Limit III* would be denoted (4, 3, 3) in this system. Escher created a Euclidean pattern in this family, his notebook drawing number 123, denoted (3, 3, 3), in which each fish swims in one of three directions. The pattern on the 2003 Math Awareness Month poster is (5, 3, 3) in this system, and is shown in Figure 11.

Figure 11. The Math Awareness Month poster.

Summary

Over a period of five decades, a series of mathematical inspirations and "shocks" have led from Coxeter's figure to the 2003 Math Awareness Month poster. Many people have been inspired by Escher's work, including the authors of articles in the recent book *M.C. Escher's Legacy*. My article and electronic file on the CDRom that accompanies that book contain a number of other examples of computer-generated hyperbolic tessellations inspired by Escher's art. I only hope that the reader has as many enjoyable inspirations and "shocks" in his or her mathematical investigations. ■

For Further Reading

There are illuminating quotes from Escher's correspondence with H. S. M. Coxeter in Coxeter's paper "The non-Euclidean symmetry of Escher's Picture 'Circle Limit III,'" *Leonardo* 12 (1979), 19–25, 32, which also shows Coxeter's calculation of the angle of intersection of the white arcs with the bounding circle in *Circle Limit III*. Read about artists who have been inspired by Escher and are currently creating new mathematical "Escher" art in the book *M. C. Escher's Legacy: A Centennial Celebration*, Doris Schattschneider and Michele Emmer, editors, Springer Verlag, 2003. *Euclidean and Non-Euclidean Geometries*, Marvin Greenberg, 3rd Edition, W. H. Freeman and Co., 1993, has a good account of the history of hyperbolic geometry and the Poincaré disk model. If you want to construct your own hyperbolic tessellation by classical methods, see "Compass and straightedge in the Poincaré disk," Chaim Goodman-Strauss, *American Mathematical Monthly*, 108 (2001), no. 1, 38-49; to do it by computer, see "Hyperbolic symmetry," Douglas Dunham, *Computers and Mathematics with Applications*, Part B 12 (1986), no. 1-2, 139-153.

Acknowledgement

I would like to thank Doris Schattschneider for her considerable help, especially with the history of Escher and Coxeter's correspondence.

Whirled White Web

Team Minnesota's (Stan Wagon, Carlo Séquin, Brent Collins, Steve Reinmuth, and Dan Schwalbe) silver-medal-winning entry in the Breckenridge, Colorado snow sculpture competition. See www.stanwagon.com for details. ❄