How to Make a Correct Multiprocess Program Execute Correctly on a Multiprocessor

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Abstract—A multiprocess program executing on a modern multiprocessor must issue explicit commands to synchronize memory accesses. A method is proposed for deriving the necessary commands from a correctness proof of the underlying algorithm in a formalism based on temporal relations among operation executions.

Index Terms—Concurrency, memory consistency, multiprocessor, synchronization, verification.

1 THE PROBLEM

Accessing a single memory location in a multiprocessor is traditionally assumed to be atomic. Such atomicity is a fiction; a memory access consists of a number of hardware actions, and different accesses may be executed concurrently. Early multiprocessors maintained this fiction, but more modern ones usually do not. Instead, they provide special commands with which processes themselves can synchronize memory accesses. The programmer must determine, for each particular computer, what synchronization commands are needed to make his program correct.

One proposed method for achieving the necessary synchronization is with a constrained style of programming specific to a particular type of multiprocessor architecture [7], [8]. Another method is to reason about the program in a mathematical abstraction of the architecture [5]. We take a different approach and derive the synchronization commands from a proof of correctness of the algorithm.

The commonly used formalisms for describing multiprocess programs assume atomicity of memory accesses. When an assumption is built into a formalism, it is difficult to discover from a proof where the assumption is actually needed. Proofs based on these formalisms, including invariance proofs [4], [16] and temporal-logic proofs [17], therefore seem incapable of yielding the necessary synchronization requirements. We derive these requirements from proofs based on a little-used formalism that makes no atomicity assumptions [11], [12], [14]. This proof method is quite general and has been applied to a number of algorithms. The method of extracting synchronization commands from a proof is described by an example—a simple mutual exclusion algorithm. It can be applied to the proof of any algorithm.

Most programs are written in higher-level languages that provide abstractions, such as locks for shared data, that free the programmer from concerns about the memory architecture. The compiler generates synchronization commands to implement the abstractions. However, some algorithms—especially within the operating system—require more efficient implementations than can be achieved with high-level language abstractions. It is to these algorithms, as well as to algorithms for implementing the higher-level abstractions, that our method is directed.

2 THE FORMALISM

An execution of a program is represented by a collection of operation executions with the two relations \( \rightarrow \) (read precedes) and \( \leftrightarrow \) (read can affect). An operation execution can be interpreted as a nonempty set of events, where the relations \( \rightarrow \) and \( \leftrightarrow \) have the following meanings.

\[ A \rightarrow B: \text{every event in } A \text{ precedes every event in } B. \]
\[ A \leftrightarrow B: \text{some event in } A \text{ precedes some event in } B. \]

However, this interpretation serves only to aid our understanding. Formally, we just assume that the following axioms hold for any operation executions \( A, B, C, \) and \( D. \)

A1. \( \rightarrow \) is transitive (\( A \rightarrow B \rightarrow C \text{ implies } A \rightarrow C \)) and irreflexive (\( A \not\leftrightarrow A \)).
A2. \( A \rightarrow B \) implies \( A \leftrightarrow B \) and \( B \leftrightarrow A \).
A3. \( A \rightarrow B \rightarrow C \text{ or } A \leftrightarrow B \rightarrow C \text{ implies } A \leftrightarrow C. \)
A4. \( A \rightarrow B \rightarrow C \rightarrow D \text{ implies } A \rightarrow D. \)
A5. For any \( A \) there are only a finite number of \( B \) such that \( A \rightarrow B. \)

The last axiom essentially asserts that all operation executions terminate; nonterminating operations satisfy a different axiom that is not relevant here. Axiom A5 is useful only for proving liveness properties; safety properties are proved with Axioms A1–A4. properties. Anger [3] and Abraham et al. [1] introduced the additional axiom

A6. \( A \leftrightarrow B \rightarrow C \rightarrow D \text{ implies } A \leftrightarrow D. \)

and showed that A1–A6 form a complete axiom system for the interpretation based on operation executions as sets of events.

Axioms A1–A6 are independent of what the operation executions do. Reasoning about a multiprocess program requires additional axioms to capture the semantics of its
processes. The appropriate axioms for read and write operations will depend on the nature of the memory system.

The only assumptions we make about operation executions are axioms A1–A5 and axioms about read and write operations. We do not assume that \( \rightarrow \) and --- \( \rightarrow \) are the relations obtained by interpreting an operation execution as the set of all its events. For example, sequential consistency [10] is equivalent to the condition that \( \rightarrow \) is a total ordering on the set of operation executions—a condition that can be satisfied even though the events comprising different operation executions are actually concurrent.

This formalism was developed in an attempt to provide elegant proofs of concurrent algorithms—proofs that replace conventional behavioral arguments with axiomatic reasoning in terms of the two relations \( \rightarrow \) and --- \( \rightarrow \). Although the simplicity of such proofs has been questioned (other assumptions.), they do tend to capture the essence of why an algorithm works.

3 An Example
3.1 An Algorithm and Its Proof

Fig. 1 shows process \( i \) of a simple \( N \)-process mutual exclusion algorithm [13]. We prove that the algorithm guarantees mutual exclusion (two processes are never concurrently in their critical sections). The algorithm is also deadlock-free (some critical section is eventually executed unless all processes halt in their noncritical sections), but we do not consider this liveness property. Starvation of individual processes is possible.

repeat forever
noncritical section;
\( l \) \( : x_i \) := true;
for \( j \) := 1 until \( i - 1 \)
do if \( x_j \) then \( x_i \) := false;
while \( x_i \) do od;
goto \( \ell \) fi od;
for \( j \) := \( i + 1 \) until \( N \) do while \( x_j \) do od od;
critical section;
\( x_i \) := false
end repeat

Fig. 1. Process \( i \) of an \( N \)-process mutual-exclusion algorithm.

The algorithm uses a standard protocol to achieve mutual exclusion. Before entering its critical section, each process \( i \) must first set \( x_i \) true and then find \( x_j \) false, for all other processes \( j \). Mutual exclusion is guaranteed because, when process \( i \) finds \( x_j \) false, process \( j \) cannot enter its critical section until it sets \( x_j \) true and finds \( x_i \) false, which is impossible until \( i \) has exited the critical section and reset \( x_i \). The proof of correctness formalizes this argument.

To prove mutual exclusion, we first name the following operation executions that occur during the \( n \)th iteration of process \( i ' s \) repeat loop.

\( L_i^m \) The last execution of statement \( l \) prior to entering the critical section. This operation execution sets \( x_i \) to \( true \).

\( R_{i,j}^m \) The last read of \( x_j \) before entering the critical section. This read obtains the value \( false \).

\( CS_i^m \) The execution of the critical section.

\( X_j^m \) The write to \( x_j \) after exiting the critical section. It writes the value \( false \).

Mutual exclusion asserts that \( CS_i^m \) and \( CS_j^m \) are not concurrent, for all \( m \) and \( n \), if \( i \neq j \). Two operations are nonconcurrent if one precedes \( \rightarrow \) the other. Thus, mutual exclusion is implied by the assertion that, for all \( m \) and \( n \), either \( CS_i^m \rightarrow CS_j^m \) or \( CS_j^m \rightarrow CS_i^m \), if \( i \neq j \).

The proof of mutual exclusion, using axioms A1–A4 and assumptions B1–B4 below, appears in Fig. 2. It is essentially the same proof as in [13], except that the properties required of the memory system have been isolated and named B1–B4. (In [13], these properties are deduced from other assumptions.)

**Theorem.** For all \( m, n, i, j \) such that \( i \neq j \), either \( CS_i^m \rightarrow CS_j^m \) or \( CS_j^m \rightarrow CS_i^m \).

Case A: \( R_{i,j}^m \rightarrow L_j^m \).

1. \( L_j^m \rightarrow R_{i,j}^m \)

   *Proof*: B1, case assumption, B1 (applied to \( L_j^m \) and \( R_{i,j}^m \)), and A4.

2. \( R_{i,j}^m \rightarrow L_j^m \)

   *Proof*: 1 and A2.

3. \( X_i^m \rightarrow R_{i,j}^m \)

   *Proof*: 2 and B4 (applied to \( R_{i,j}^m \) and \( L_j^m \) and \( X_i^m \)).

4. \( CS_i^m \rightarrow CS_j^m \)

   *Proof*: B3, 3, B2 (applied to \( R_{i,j}^m \) and \( CS_j^m \)), and A4.

Case B: \( R_{i,j}^m \rightarrow L_j^m \).

1. \( X_j^m \rightarrow R_{i,j}^m \)

   *Proof*: Case assumption and B4.

2. \( CS_j^m \rightarrow CS_i^m \)

   *Proof*: B3 (applied to \( CS_j^m \) and \( X_j^m \)), 1, B2, and A4.

Fig. 2. Proof of mutual exclusion for the algorithm of Fig. 1.

B1–B4 are as follows, where universal quantification over \( n, m, i, \) and \( j \) is assumed. B4 is discussed below.

B1. \( L_i^m \rightarrow R_{i,j}^m \)

B2. \( R_{i,j}^m \rightarrow CS_i^m \)

B3. \( CS_i^m \rightarrow X_i^m \)

B4. If \( R_{i,j}^m \rightarrow L_j^m \) then \( X_j^m \) exists and \( X_j^m \rightarrow R_{i,j}^m \).

Although B4 cannot be proved without additional assumptions, it merits an informal justification. The hypothesis,

1. Except where indicated otherwise, all assertions have an unstated hypothesis the assumption that the operation executions they mention actually occur. For example, the theorem in Fig. 2 has the hypothesis that \( CS_i^m \) and \( CS_j^m \) occur.
that the value obtained by the read was written by process preceding the critical section is not needed. However, B1–B3 are prece- dence relations between operations issued by the same proc-

3.2 The Implementation
Implementing the algorithm for a particular memory ar-

3.3 Observations
One might think that the purpose of memory synchroniza-

4 Further Remarks
The atomicity condition traditionally assumed for multi-

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Assertional proofs are practical for more complicated algorithms. The obvious way to reason assertionally about algorithms with nonatomic memory operations is to represent a memory access by a sequence of atomic operations \[2, 9\]. With this approach, the memory architecture and synchronization operations are encoded in the algorithm. Therefore, a new proof is needed for each architecture, and the proofs are unlikely to help discover what synchronization operations are needed. A less obvious approach uses the predicate transformers \(\text{win}\) (weakest invariant) and \(\text{sin}\) (strongest invariant) to write assertional proofs for algorithms in which no atomic operations are assumed, requirements on the memory architecture being described by axioms \[15\]. Such a proof would establish the correctness of an algorithm for a large class of memory architectures. However, in this approach, all intraprocess \(\rightarrow\) relations are encoded in the algorithm, so the proofs are unlikely to help discover the very precedence relations that lead to the introduction of synchronization operations.

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**REFERENCES**


