Probabilistic Bounds on the Length of a Longest Edge in Delaunay Graphs of Random Points in $d$-Dimensions

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The Problem

- Longest Delaunay edge in multidimensional Euclidean spaces.
- Multidimensional body of volume 1.
- Set of points distributed uniformly at random in it.

Motivation:
- Length of longest edge: bound communication cost in wireless nets.
- Points deployment: RGG’s.

Length of longest Delaunay edge strongly influenced by boundaries
⇒ we study enclosing bodies
  (i) with boundary (e.g. disk).
  (ii) without boundary (e.g. sphere (ball surface)).
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Previous Work

- **Longest Delaunay edge**
  - inside a hypercube of size $n$ (infinite Poisson point set)
  - Bern, Eppstein, Yao (JCGA 1991): observe $\Theta\left(\sqrt[3]{\log n}\right)$ in expectation. asymptotic only, expectation.
  - RGG:
    - Kozma, Lotker, Sharir, Stupp (PODC 2004):
      $O\left(\frac{3}{\sqrt[3]{\log n}}\right)$ w.h.p. for points “close” to boundary.
      $O\left(\frac{1}{\sqrt{\log n}}\right)$ w.h.p. for points “away” from boundary.
      $d = 2$ only, asymptotic only, fixed error probability.

- **Longest Gabriel edge**:
  - Wan, Yi (TPDS 2007): show $\leq 2\sqrt{(\ln n)/(\pi n)}$ a.a.s. for $d = 2$.
  - Devroye, Gudmundsson, Morin (arXiv 2009): observe $O\left(\frac{d}{\sqrt{\log n}}\right)$ a.a.s.

- **Multidimensional Delaunay tessellations**: (construction algorithms)
  - Devijver, Dekesel (PRL 1983)
  - Lemaire, Moreau (CG 2000)
Our results

Upper and lower bounds
for $d$-dimensional bodies,
with and without boundaries,
with parametric error probability $\varepsilon$,
and up to constants.

- Tight for $e^{-cn} \leq \varepsilon \leq n^{-c}$ and $d \in O(1)$.
- For $d = 2$ and $\varepsilon = 1/n$, UB matches [KLSS 04].
- First comprehensive study of this problem.
  (LBs with boundary for $d \in \{2, 3\}$.)
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Preliminaries

Definition
Let $P$ be a set of points in a $d$-sphere, two points $a, b \in P$ form an arc of $D(P)$, if and only if there is a $d$-dimensional spherical cap $C$ such that, with respect to the surface of the cap, it contains $a$ and $b$ on the boundary and does not contain any other point of $P$. 
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Let $P$ be a set of points in a $d$-ball, two points $a, b \in P$ form an edge of $D(P)$, if and only if there is a $d$-ball $B$ such that, $a$ and $b$ are located in the surface area of $B$, and the interior of $B$ does not contain any other point of $P$. 
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Results

Proof techniques

- Upper bounds: thanks to uniform density,
  a “large” empty area/volume is “unlikely”.
- Lower bounds: show configuration such that
  “long” Delaunay edge is “not very unlikely”.

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Longest Delaunay Edges
Results

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- Lower bounds: show configuration such that “long” Delaunay edge is “not very unlikely”.

For enclosing bodies with boundaries...

... witness $d$-ball may be huge!
Results

Without boundary

<table>
<thead>
<tr>
<th>$d$</th>
<th>$A_d(\delta(a, b)) \geq \frac{\ln\left(\frac{n \choose 2}{n-2} / \varepsilon\right)}{n-d-1}$</th>
<th>$\delta(a, b) \geq \frac{\ln\left(\frac{n \choose 2}{n-2} / \varepsilon\right)}{n-2}$</th>
<th>$\delta(a, b) \geq \cos^{-1}\left(1 - \frac{2 \ln\left(\frac{n \choose 2}{n-2} / \varepsilon\right)}{n-3}\right)$</th>
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<td>1</td>
<td>$\frac{\ln\left(\frac{n \choose 2}{n-2} / \varepsilon\right)}{n-2}$</td>
<td>$\frac{\ln\left((e-1)/(e^2 \varepsilon)\right)}{n-2 + \ln\left((e-1)/(e^2 \varepsilon)\right)}$</td>
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<tr>
<td>2</td>
<td>$\frac{\ln\left(\frac{n \choose 2}{n-2} / \varepsilon\right)}{\sqrt{\pi}}$</td>
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w.p. $\geq 1 - \varepsilon$, $\not\exists \hat{a} b \in D(P)$

w.p. $\geq \varepsilon$, $\exists \hat{a} b \in D(P)$
## Analysis

### Results

**Without boundary**

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<td>1</td>
<td>$A_d(\delta(a, b)) \geq \frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}$</td>
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<td>$\delta(a, b) \geq \cos^{-1}\left(1 - \frac{2\ln\left(\binom{n}{2}(n-2)/\varepsilon\right)}{n-3}\right)$</td>
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**UB for** $d = 2 \in \Theta\left(\sqrt{\frac{\ln(n/\varepsilon)}{n}}\right)$

... matching KLSS’04 for $\varepsilon = 1/n$. 

![Diagram](image_url)
**Results**

*With boundary*

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<td>$d$</td>
<td>$V_d(d(a, b)) \geq \frac{\ln\left(\frac{n}{2}\left(\frac{n-2}{d-1}\right)/\varepsilon\right)}{n-d-1}$</td>
<td>$d(a, b) \geq \rho_2/2 : V_2(\rho_2) = \frac{\ln(\alpha_2/\varepsilon)}{(n-2+\ln(\alpha_2/\varepsilon))}$</td>
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Results

E.g. Proof of Lower Bound in a Disk

\[ V(\rho) + \frac{1}{n} \]
Results
E.g. Proof of Lower Bound in a Disk

\[ \leq V(\rho) + \frac{1}{n} \]
Results

E.g. Proof of Lower Bound in a Disk

\[ Pr(\exists a) \in \Omega(1) \]

\[ Pr(\exists b) \in \Omega(1) \]

\[ Pr(V(\rho) + 1/n \text{ is empty}) \geq \varepsilon \]
Results

E.g. Lower Bound in a Ball

\[ \rho_1 / \sqrt{2} \]

\[ \rho_1 / 2 \]

\( h_1 \)

\( h_2 \)

\( \Gamma \)

\( \Gamma_1 \)

\( \Gamma_2 - \Gamma_1 \)
Open Problems

- Lower bound with boundary for $d > 3$?
  Conjecture: same bound modulo a constant.
- Other norms? ($L_1, L_\infty$)
- Other distribution of points.
Thank you!

Questions?