Probabilistic Bounds on the Length of a Longest Edge in Delaunay Graphs of Random Points in d-Dimensions

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The Problem

- Longest Delaunay edge in multidimensional Euclidean spaces.
- Multidimensional body of volume 1.
- Set of points distributed uniformly at random in it.

- Motivation:
  - Length of longest edge: bound communication cost in wireless nets.
  - Points deployment: RGG's.

- Length of longest Delaunay edge strongly influenced by boundaries

  $\Rightarrow$ we study enclosing bodies

  (i) with boundary (e.g. disk).

  (ii) without boundary (e.g. sphere (ball surface)).

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Previous Work

- Longest Delaunay edge
  - inside a hypercube of size n (infinite Poisson point set)
    - Bern, Eppstein, Yao (JCGA 1991): observe $\Theta\left(\sqrt[d]{\log n}\right)$ in expectation. asymptotic only, expectation.
  - RGG:
    - Kozma, Lotker, Sharir, Stupp (PODC 2004):
      $O\left(\sqrt[3]{(\log n)/n}\right)$ w.h.p. for points "close" to boundary.
      $O\left(\sqrt{(\log n)/n}\right)$ w.h.p. for points "away" from boundary.
      $d = 2$ only, asymptotic only, fixed error probability.
- Longest Gabriel edge :
  - Wan, Yi (TPDS 2007): show $\leq 2\sqrt{(\ln n)/(\pi n)}$ a.a.s. for $d = 2$.
  - Devroye, Gudmundsson, Morin (arXiv 2009): observe $O(\sqrt[d]{(\log n)/n})$ a.a.s.
- Multidimensional Delaunay tessellations: (construction algorithms)
  - Devijver, Dekesel (PRL 1983)
  - Lemaire, Moreau (CG 2000)

Our results

Upper and lower bounds

for d-dimensional bodies,

with and without boundaries,

with parametric error probability ε,

and up to constants.

- Tight for $e^{-cn} \leq \varepsilon \leq n^{-c}$ and $d \in O(1)$.
- For $d = 2$ and $\varepsilon = 1/n$, UB matches [KLSS 04].
- First comprehensive study of this problem.

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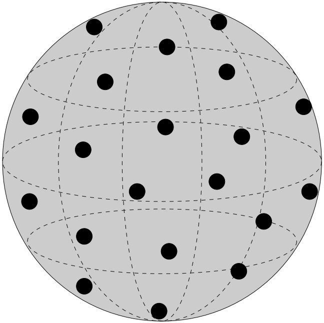
Preliminaries

Definition

Let P be a set of points in a d-sphere, two points $a, b \in P$ form an arc of $D(P)$, if and only if there is a d-dimensional spherical cap C such that, with respect to the surface of the cap, it contains a and b on the boundary and does not contain any other point of P.



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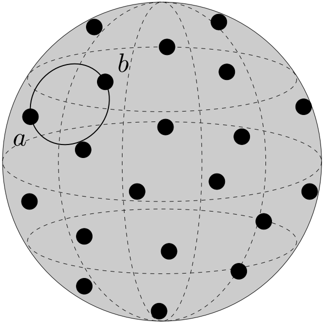

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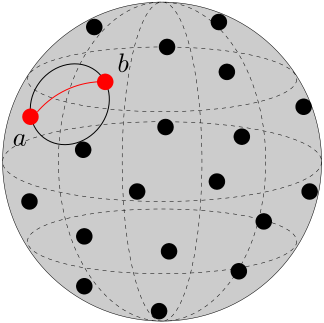

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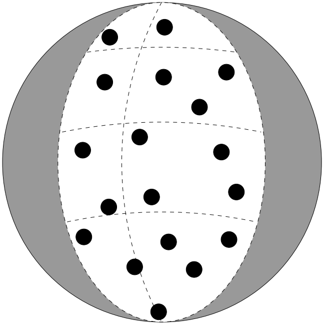

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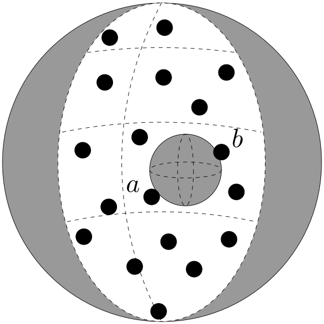

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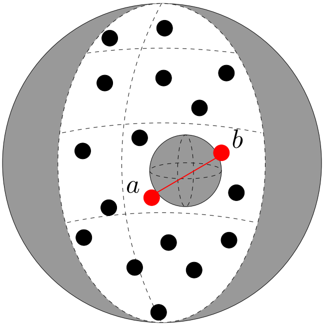

Results

Proof techniques

- Upper bounds: thanks to uniform density,
  a "large" empty area/volume is "unlikely".
- Lower bounds: show configuration such that
  "long" Delaunay edge is "not very unlikely".

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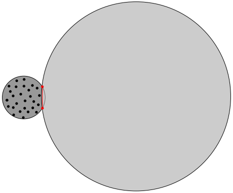
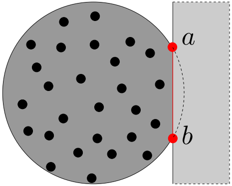
For enclosing bodies with boundaries...



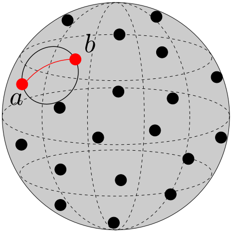


... witness d-ball may be huge!

Results

Without boundary

	w.p. $\geq 1-\varepsilon$, $\nexists\ \widehat{ab} \in D(P)$	w.p. $\geq \varepsilon$, $\exists\ \widehat{ab} \in D(P)$
d	$A_d(\delta(a,b)) \geq \frac{\ln\left(\binom{n}{2}\binom{n-2}{d-1}/\varepsilon\right)}{n-d-1}$	$A_d(\delta(a,b)) \geq \frac{\ln\left((e-1)/(e^2\varepsilon)\right)}{n-2+\ln\left((e-1)/(e^2\varepsilon)\right)}$
1	$\delta(a,b) \geq \frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}$	$\delta(a,b) \geq \frac{\ln\left((e-1)/(e^2\varepsilon)\right)}{n-2+\ln\left((e-1)/(e^2\varepsilon)\right)}$
2	$\delta(a,b) \geq \frac{\cos^{-1}\left(1-\frac{2\ln\left(\binom{n}{2}(n-2)/\varepsilon\right)}{n-3}\right)}{\sqrt{\pi}}$	$\delta(a,b) \geq \frac{\cos^{-1}\left(1-\frac{2\ln\left((e-1)/(e^2\varepsilon)\right)}{n-2+\ln\left((e-1)/(e^2\varepsilon)\right)}\right)}{\sqrt{\pi}}$



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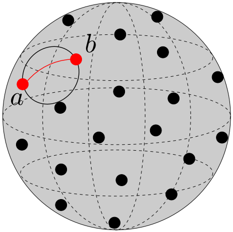
UB for $d = 2 \in \Theta\left(\sqrt{\ln\left(n/\varepsilon\right)/n}\right)$
... matching KLSS'04 for $\varepsilon = 1/n$.



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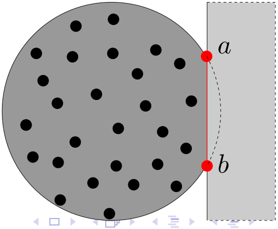
	w.p. $\geq 1-\varepsilon$, $\nexists\ \widehat{ab} \in D(P)$	w.p. $\geq \varepsilon$, $\exists\ \widehat{ab} \in D(P)$
d	$V_d(d(a,b)) \geq \frac{\ln\left(\binom{n}{2}\binom{n-2}{d-1}/\varepsilon\right)}{n-d-1}$	?
2	$d(a,b) \geq \sqrt[3]{\frac{16}{\sqrt{\pi}} \frac{\ln\left(\binom{n}{2}(n-2)/\varepsilon\right)}{n-3}}$	$d(a,b) \geq \rho_2/2 : V_2(\rho_2) = \frac{\ln(\alpha_2/\varepsilon)}{(n-2+\ln(\alpha_2/\varepsilon))}$ $\Longrightarrow d(a,b) \geq \sqrt[3]{\frac{\ln(\alpha_2/\varepsilon)}{2\sqrt{\pi}(n-2+\ln(\alpha_2/\varepsilon))}}$
3	$d(a,b) \geq \sqrt[4]{\frac{96}{\pi^{3/2}} \frac{\ln\left(\binom{n}{2}\binom{n-2}{2}/\varepsilon\right)}{n-4}}$	$d(a,b) \geq \rho_3/2 : V_3(\rho_3) = \frac{\ln(\alpha_3/\varepsilon)}{(n-2+\ln(\alpha_3/\varepsilon))}$ $\Longrightarrow d(a,b) \geq \sqrt[4]{\sqrt[3]{\frac{48}{\pi^4}} \frac{\ln(\alpha_3/\varepsilon)}{(n-2+\ln(\alpha_3/\varepsilon))}}$



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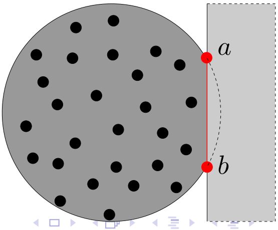

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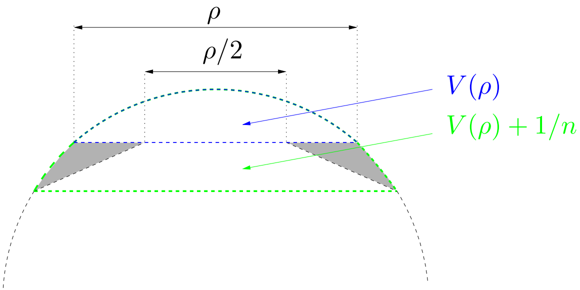
E.g. Proof of Lower Bound in a Disk



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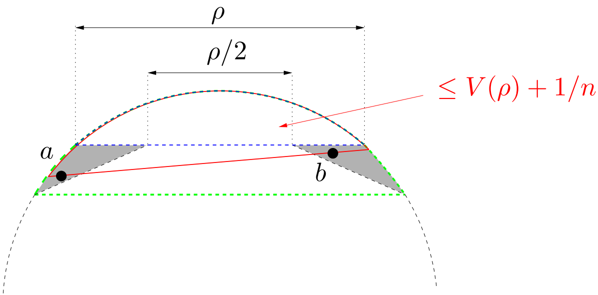

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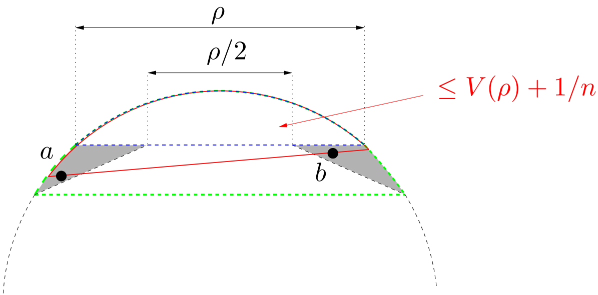
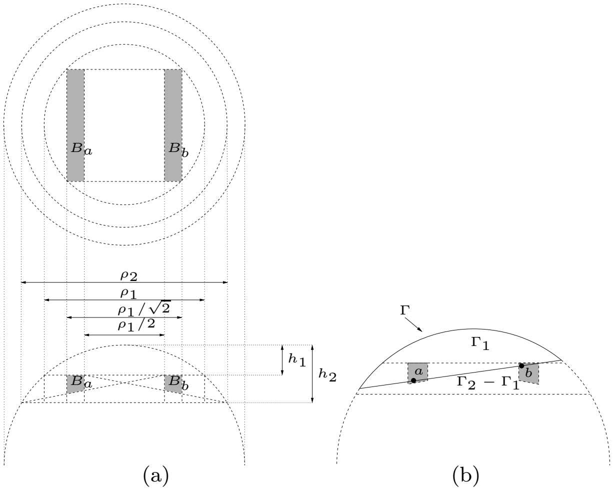


$$Pr(\exists a) \in \Omega(1)$$
$$Pr(\exists b) \in \Omega(1)$$
$$Pr(V(\rho) + 1/n \text{ is empty}) \geq \varepsilon$$

Results

E.g. Lower Bound in a Ball



Open Problems

- Lower bound with boundary for $d > 3$?

  Conjecture: same bound modulo a constant.
- Other norms? (L_1, L_∞)
- Other distribution of points.

Thank you!

Questions?