The Length of the Longest Edge in Multi-dimensional Delaunay Graphs

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The Problem

- Longest Delaunay edge in multidimensional Euclidean spaces.
- Multidimensional body of volume 1.
- Set of points distributed uniformly at random in it.
  (motivation: RGG's)

- Length of longest Delaunay edge strongly influenced by boundaries
  $\Rightarrow$ we study enclosing bodies
  (i) with boundary (e.g. disk).
  (ii) without boundary (e.g. sphere (ball surface)).

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Previous Work

- Longest Delaunay edge in $\mathbb{R}^2$:
  - Kozma, Lotker, Sharir, Stupp, PODC'04:

    $O\left(\sqrt[3]{\log n/n}\right)$ w.h.p. for points "close" to boundary.

    $O\left(\sqrt{\log n/n}\right)$ w.h.p. for points "away" from boundary.

- Multidimensional Delaunay tessellations:
  - Devijver, Dekesel, PRL, 1983.
  - Lemaire, Moreau, CG, 2000.

  Construction algorithmic techniques.

Our results

Upper and lower bounds

for d-dimensional bodies, $d > 1$

with and without boundaries,

with parametric error probability ε,

and up to constants.

- Tight for $e^{-cn} \leq \varepsilon \leq n^{-c}$, for constant $c > 0$.
- UB matches [KLSS 04] for $d = 2$, $\varepsilon = 1/n$.
- First comprehensive study of this problem.

  (LBs with boundary for $d \in \{2, 3\}$.)

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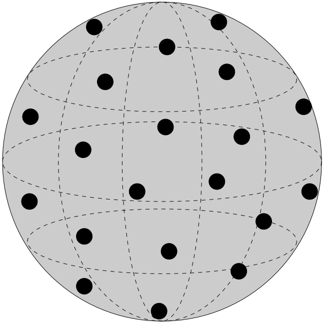
Preliminaries

Definition

Let P be a set of points in a d-sphere, two points $a, b \in P$ form an arc of $D(P)$, if and only if there is a d-dimensional spherical cap C such that, with respect to the surface of the cap, it contains a and b on the boundary and does not contain any other point of P.



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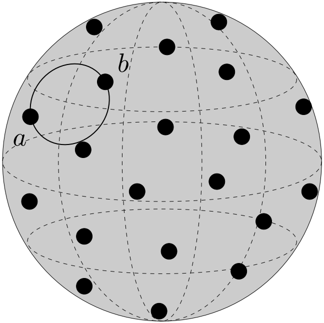

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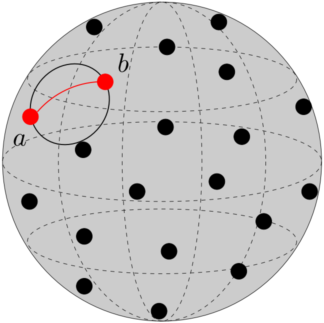

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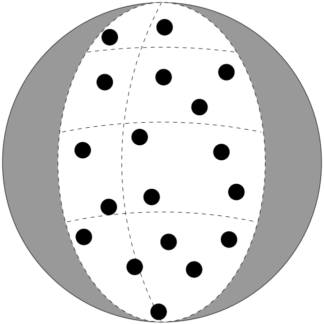
Definition

Let P be a set of points in a d-ball, two points $a, b \in P$ form an edge of $D(P)$, if and only if there is a d-ball B such that, a and b are located in the surface area of B, and the interior of B does not contain any other point of P.



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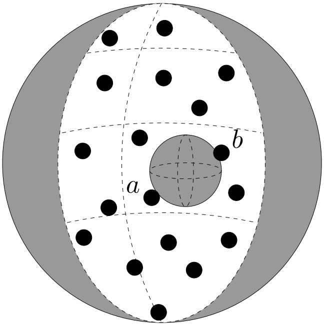

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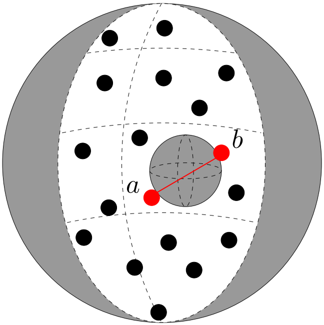

Results

Proof techniques

- Upper bounds: thanks to uniform density,

  a “large” empty area/volume is “unlikely”.
- Lower bounds: show configuration such that

  “long” Delaunay edge is “not very unlikely”.

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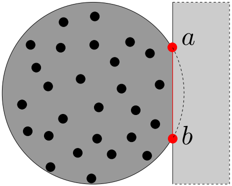
For enclosing bodies with boundaries...

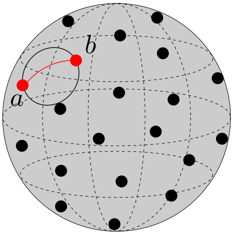


... witness d-ball may be huge!

Results

Without boundary

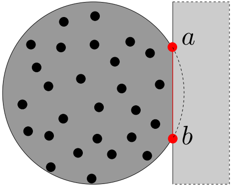
d	$\nexists\ \widehat{ab} \in D(P)$ w.p. $\geq 1-\varepsilon$	$\exists\ \widehat{ab} \in D(P)$ w.p. $\geq \varepsilon$
d	$A_d(1,\delta(a,b)) \geq \frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}$	$A_d(1,\delta(a,b)) \geq \frac{\ln\left((e-1)/(e^2\varepsilon)\right)}{n-2+\ln\left((e-1)/(e^2\varepsilon)\right)}$
2	$\delta(a,b) \geq \frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}$	$\delta(a,b) \geq \frac{\ln\left((e-1)/(e^2\varepsilon)\right)}{n-2+\ln\left((e-1)/(e^2\varepsilon)\right)}$
3	$\delta(a,b) \geq \frac{\cos^{-1}\left(1-\frac{2\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}\right)}{\sqrt{\pi}}$	$\delta(a,b) \geq \frac{\cos^{-1}\left(1-\frac{2\ln\left((e-1)/(e^2\varepsilon)\right)}{n-2+\ln\left((e-1)/(e^2\varepsilon)\right)}\right)}{\sqrt{\pi}}$



Results

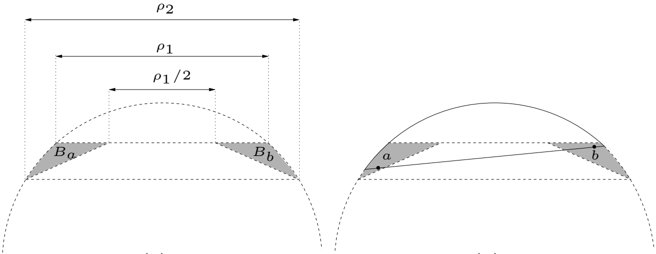
With boundary

	$\nexists\ \widehat{ab} \in D(P)$ w.p. $\geq 1-\varepsilon$	$\exists\ \widehat{ab} \in D(P)$ w.p. $\geq \varepsilon$
d	$V_d(1, \lvert\lvert\overrightarrow{a,b}\rvert\rvert_2) \geq \frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}$	–
2	$\lvert\lvert\overrightarrow{a,b}\rvert\rvert_2 \geq \sqrt[3]{\frac{16}{\sqrt{\pi}} \frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}}$	$\lvert\lvert\overrightarrow{ab}\rvert\rvert_2 \geq \rho_2/2 : V_d(1,\rho_2) = \frac{\ln(\alpha_2/\varepsilon)}{(n-2+\ln(\alpha_2/\varepsilon))}$
3	$\lvert\lvert\overrightarrow{a,b}\rvert\rvert_2 \geq \sqrt[4]{\frac{96}{\pi^{3/2}} \frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}}$	$\lvert\lvert\overrightarrow{ab}\rvert\rvert_2 \geq \rho_3/2 : V_d(1,\rho_3) = \frac{\ln(\alpha_3/\varepsilon)}{(n-2+\ln(\alpha_3/\varepsilon))}$



Results

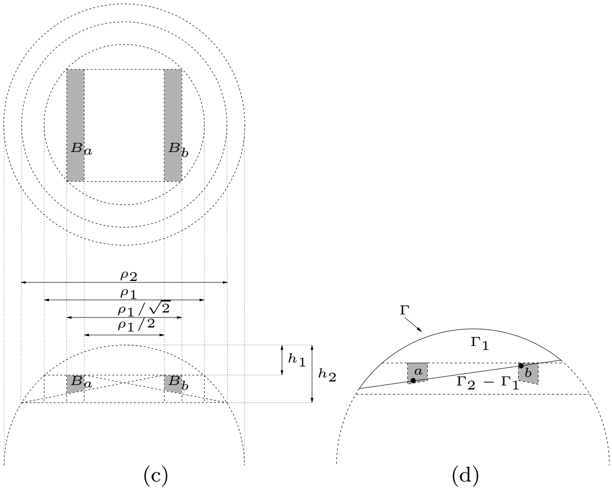
E.g. Lower Bound in a Disk



(a) (b)

Results

E.g. Lower Bound in a Ball



Open Problems

- Other norms? (L_1, L_∞)
- Lower bound with boundary for $d > 3$?

  Conjecture: same bound modulo a constant.

Thank you