THE START

QM Background

See Slides 30 & 31 for the difference in emphasis in Quantum Computing versus Quantum Mechanics.

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QM Background [0]

An Hermitian Operator (matrix) has all the properties we need to model the quantum thingies of interest. $H = H^{\dagger} = H^{*T}$

1. It has a complete spanning set of eigenvectors. $\underline{v}_i \ni H\underline{v}_i = \lambda_i \underline{v}_i$
   - They can be orthonormal-ized.
2. Its eigenvalues are always real. λ_i
3. It is self adjoint so it allows of the spectral decomposition.

$$H = \sum_{i=1}^{i=n} \lambda_i \underline{v}_i \underline{v}_i^T$$

4. $\underline{v}_i \underline{v}_i^T \triangleq P_{\underline{v}_i}$ Is the projector on the eigenvector $\underline{v}_i$

QM Background [1]

1. A QM Thingie of Interest (TOI), like energy or position, or momentum, or a photon's state of polarization, corresponds to an operator that mushes around the vectors which are the measureable states of the Thingie.

   {Operator ←-> Thingie}
   &
   {Thingie state ←> vector}

2. A Quantum System in a physical state is represented by a corresponding UNIT vector in some ~~abstract~~ Hermitian vector space.

QM Background [2/6]

3. A Measurement puts the Quantum System into a unique physical state called a "Pure State" represented by a vector along a **UNIT** Basis Vector in that ~~abstract~~ vector space.

4. Before any measurement, the system is in an **unknown** mixture of pure states, called a "Mixed State Vector".

5. A measurement corresponds to a **projection** of a UNIT mixed vector onto ONE of the UNIT basis vectors of the ~~abstract~~ space. "Pure State Vector".

QM Background [3/6]

6. The **(length)^2 <1** of the projected unit mixed state vector (the pure state vector) is the **PROBABILITY** of finding that Pure State in any given measurement.

   {Projected pure state length^2 = Probability of getting that measurement result}

7. All Basis Vectors are actually eigenvectors of the operator representing the measured quantity. {Possible Pure States}

QM Background [4/6]

8. The **value of the eigenvalue** corresponding to the measured pure state is the **measured VALUE** of the Thingie in that pure state.
   {Eigenvalue of Projected pure state =
   Value of measurement result.}

QM Background [5/6]

Outer Product as a Projector

$$\begin{bmatrix} \\ \\ \end{bmatrix}\begin{bmatrix} & & \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \approx V_i V_i^* = P_i$$

Applied to Projections, regroup.

$$\left\{\begin{bmatrix} \\ \\ \end{bmatrix}\begin{bmatrix} & & \end{bmatrix}\right\}\begin{bmatrix} \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix}\left\{\begin{bmatrix} & & \end{bmatrix}\begin{bmatrix} \\ \\ \end{bmatrix}\right\}$$

MATRIX-Vector=Vector-Scalar

$$\left\{V_i V_i^*\right\} V_i = PV_i = V_i = V_i\left\{V_i^* V_i\right\} = V_i\left(1\right) = V_i$$ Projector on eigenvector

$$\left\{V_i V_i^*\right\} V_{j \neq i} = PV_{j \neq i} = V_i\left\{V_i^* V_{j \neq i}\right\} = V_i\left(0\right) = \underline{0}$$ Orthogonal Projector

QM Background [6/6]

9. A measurement corresponds to a meter reading of a physical system in an unknown state yielding a known state with a known metered value.

Operator	Thingie (Property)
Vector $\underline{m}$	Unknown State of Thingie
Projected Vector $\underline{v}_i$ is an Eigenvector of Operator	Known State
Eigenvector Length ^{2} = sum $(\alpha_i)^2$	Prob of getting state
Eigenvalue of Eigenvector λ_i	Value of Thingie

Eigen Value & Vector

$$H\underline{v} = \lambda \underline{v}$$

$$\underline{m} = \Sigma\, \alpha_i(\, \underline{v}_i)$$

Mixed State

$$H = \Sigma\, \lambda_i(\, \underline{v}_i\, \underline{v}_i^*)$$

Spectral Decomposition of a Normal (Hermitian) Operator as weighted sum of projectors.

$$H\, \underline{v}_i = \Sigma\, \lambda_i([\underline{v}_i\, \underline{v}_i^*]\, \underline{v}_i\,) = \lambda_i[\underline{v}_i](1) = \lambda_i \underline{v}_i$$

QM Background [7]

1. A vector space is a collection of thingies that **add** ($\underline{u}=\underline{v}+\underline{w}$), **associate** $\underline{u}+(\underline{w}+\underline{z})=(\underline{u}+\underline{w})+\underline{z}$, have an **identity** ($\underline{u}+\underline{0}=\underline{u}$), an **additive inverse** ($-\underline{u}+\underline{u}=\underline{0}$), and **commute (+)** ($\underline{w}+\underline{z})=(\underline{z}+\underline{w}$).
2. There is also a field of **scalars** that multiply them: $k\underline{u} \Rightarrow (|k||\underline{u}|)$. This is scalar multiplication. Scalars here are **<u>complex #s</u>**.
3. In addition to this scalar multiplication, there is a (vector·vector) multiplication called the <u>inner product</u> (= the scalar product = the dot product). It yields a <u>scalar</u> and is notated **<u>u</u>·<u>v</u>** [“<u>u</u> dot <u>v</u>”] .

QM Background [8]

"Hermitian" Inner Product.

3\. a. $\langle -, - \rangle$ **[maps to vectors to a scalar (COMPLEX NUMBER)]**

3\. b. $\langle \underline{u}+\underline{v}, \underline{w} \rangle = \langle \underline{u}, \underline{w} \rangle + \langle \underline{v}, \underline{w} \rangle$ [$\underline{u}$, $\underline{v}$, $\underline{w}$ are vectors]

3\. c. $\langle \underline{u}, \underline{v}+\underline{w} \rangle = \langle \underline{u}, \underline{v} \rangle + \langle \underline{u}, \underline{w} \rangle$ Distribution

3\. d. $\langle a\underline{u}, \underline{v} \rangle = a\langle \underline{u}, \underline{v} \rangle$ [a is a COMPLEX NUMBER]

Linear in the left position

3\. e. $\langle \underline{u}, a\underline{v} \rangle = a^*\langle \underline{u}, \underline{v} \rangle$ [* is COMPLEX CONJUGATION]

Conjugate Linear in the right position

3\. f. $\langle \underline{u}, \underline{v} \rangle = \langle \underline{v}, \underline{u} \rangle^*$ [* is COMPLEX CONJUGATION]

3\. g. $\langle \underline{u}, \underline{u} \rangle = |\underline{u}|^2 \geq 0$ [= 0 iff $\underline{u} = \underline{0}$]

[Eric W. Weisstein et al. "Hermitian Inner Product." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/HermitianInnerProduct.html]

If $\underline{u}$ and $\underline{v}$ are real: $\langle \underline{u}, \underline{v} \rangle = \langle \underline{v}, \underline{u} \rangle =$ def $(\underline{u}\cdot\underline{v})$

$\underline{u}\cdot\underline{v} = |\underline{u}||\underline{v}|\cos(\text{angle between them})$.

QM Background [9]

4. |u|= (u·u*)$^{.5}$ is a real number. "Length"
5. Hermitian operators [H=H*] (* is conjugate transpose) map vectors to vectors in the vector space. Hu = v .
6. Eigenvectors ("ownvectors") of an operator H are those vectors that H maps into **multiples of themselves.** Hu=λu . If |u| =1, λ is an eigenvalue of H associated with u [there can be more than one u for a given λ].
7. An Hermitian operator's eigenvalues are **real (λ)**.
8. An Hermitian operator's eigenvectors form a **basis** of the entire [Hermitian Vector] space.
9. In a real inner product space the symmetric operators ($A=A^t$) are the Hermitian operators.

QM Background [10]

Why “EIGEN”?

Let $\underline{u}_i$ be the eigen vectors of H

and

Let λ_i be the corresponding eigen values.

Then

The “Spectral” Decomposition of the Operator H. The λ_i are the “spectra”.

$$H = \Sigma \lambda_i (\underline{u}_i \underline{u}_i^*)$$

“Rank 1 decomposition” of numerical analysis

Notice that $(\underline{u}_i \underline{u}_i^*)$ acts as a projector operator onto $\underline{u}_i$.

$(\underline{u}_i \underline{u}_i^*)\underline{v} = \underline{u}_i (\underline{u}_i^* \underline{v}) = \underline{u}_i (\underline{\text{const.}})$. $(\underline{u}_i \underline{u}_i^*)$ is NOT $\langle \underline{u}_i, \underline{u}_i^* \rangle$ it is the juxtaposition of two vectors.

$$\begin{bmatrix} x \\ x \\ x \\ x \end{bmatrix} [x, x, x, x] = \begin{pmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{pmatrix}$$

$$[x, x, x, x] \begin{bmatrix} x \\ x \\ x \\ x \end{bmatrix} = (\text{SCALAR})$$

QM Background [11]

Quantum Mechanics is "just" modeling a physical system by "the right" Hermitian vector space.

Measurable Quantity ⟷ [$H^* = H$] ⟷ Hermitian Operator

Measured Value ⟷ [$H\underline{v} = \lambda\underline{v}$] ⟷ Eigenvalue λ

[Sometimes we don't care about the values!]

Measured State ["Pure"] ⟷ Eigenvector

[Sometimes we know the eigenvectors so we don't need the operator!]

[Most times we care only about the line, not the +- direction!] [$H(-\underline{v}) = \lambda(-\underline{v})$]

Gen. System States ⟷ Eigenvector Combination

↓ ↓

Probability of Value ⟷ (Length)^2 of projection on resultant eigenvector [<=1]

[All states = length 1, all eigenvalues real]

QM Background [12]

1. A Quantum System in a physical state is represented by a corresponding UNIT vector in some ~~abstract~~ Hermitian vector space.
2. A **Measurement** puts the Quantum System into a unique physical state called a "**Pure State**" represented by a vector along a **UNIT Basis Vector** in that ~~abstract~~ vector space.
3. Before any measurement, the system is in an <u>unknown</u> mixture of pure states, called a "**Mixed State**".
4. A measurement corresponds to a **projection** of a UNIT mixed vector onto ONE of the UNIT basis vectors of the ~~abstract~~ space.

QM Background [13]

5. The **(length)²** <1 of the projected unit mixed state vector is the **PROBABILITY** of finding that Pure State in any given measurement.
6. All **Basis Vectors** are actually **eigenvectors** of the operator representing the measured quantity.
7. The value of the **eigenvalue** corresponding to the pure state is the measured **VALUE** in that pure state.
8. A measurement corresponds to a meter reading of a physical system in an unknown state yielding a known state with a known metered value.

QM Background [14] Definitions

$H \triangleq$ Operator (moves vectors to vectors)

v $\triangleq$ Vector (has a lenght and a direction) (a sum of basis vectors)

$\lambda \triangleq$ Eigenvalue (of H) iff Hv = λv

in which case, v is called an Eigenvector associated with λ

(H just 'stretches' v but doesn't move it)

|v| $\triangleq$ length of v. |v|=1, v is a unit vector.

$H \triangleq H^{*T}$ (conjugate transpose)

$H = H^{\dagger}$ then H is called "Hermitian" [It has only real eigenvalues]

and is has a complete set of basis vectors (ANY vector is a weighted sum of the basis vectors)

<v,w> is an Hermitian inner product if it is a complex number,

$\langle v,w\rangle = \langle w,v\rangle^{*}$ and $\langle \alpha u+\beta v,w\rangle = \alpha\langle u,w\rangle+\beta\langle v,w\rangle$

and $\alpha\langle u,w\rangle = \alpha^{*}\langle w,u\rangle$ i.e., it is "conjugate linear"

QM Background [15] Definitions

$H \equiv \lambda_1 \mathrm{v}\,\mathrm{v}^* + \lambda_2 \mathrm{u}\,\mathrm{u}^*$ $\quad \mathrm{u}^*\mathrm{u} \triangleq <\mathrm{u}^*, \mathrm{u}>=1\,; \mathrm{v}^*\mathrm{v} \triangleq <\mathrm{v}^*, \mathrm{v}>=1; <\mathrm{u}^*, \mathrm{v}>=0$

$\mathrm{u}\,\mathrm{u}^*$ is an operator (matrix)

$H\mathrm{v} = \lambda_1 \mathrm{v}\,(\mathrm{v}^*\mathrm{v}) + \lambda_2 \mathrm{u}\,(\mathrm{u}^*\mathrm{v}) = \lambda_1 \mathrm{v}\,(1) + \lambda_2 \mathrm{u}\,(0) = \lambda_1 \mathrm{v}$ An Eigenvector

$U = U^* \quad UU^* = U^*U = I$ Unitary Operator .

$< \mathrm{u}^* U^*, U\mathrm{u}> = < \mathrm{u}^*, U^*U\mathrm{u}> = < \mathrm{u}^*, I\mathrm{u}>=1$ Preserves lengths (like a rotation)

UNITARY EQUIVALENCE (Rotate the *Operator* & *vectors*)

$$\hat{\mathrm{H}} = \mathrm{U}H\mathrm{U}^* = \lambda_1 (\mathrm{Uv})\,(\mathrm{v}^*\mathrm{U}^*) + \lambda_{2U} (\mathrm{Uu})\,(\mathrm{u}^*\mathrm{U}^*) = \lambda_1 \left(\hat{\mathrm{v}}\right)\left(\hat{\mathrm{v}}\right)^* + \lambda_2 \left(\hat{\mathrm{u}}\right)\left(\hat{\mathrm{u}}\right)^*$$

where $\hat{\mathrm{v}} = \mathrm{Uv}$ $\quad \hat{\mathrm{u}} = \mathrm{Uu}$

QM Background [16]

Anything we can measure is *represented* by an *operator* on a Hermitian Inner Product Vector Space.

QM Background [17]

An operator represents the thingie being measured, like Photon Polarization

QM Background [18]

The STATE of the thingie is an ***UNKNOWN*** messy mix of weighted basis vectors (possibilities) (eigenvectors) *before* the measurement.

QM Background [19]

The measurement *projects* the mess onto a single eigenvector (a **"pure"** state).

QM Background [20]

The ***length***^{2} of the projection is the **probability** of getting this eigenvector (“pure” state).

QM Background [21]

Gimmick: All states are unit length. SO, all projections are ≤ 1

QM Background [22]

≤ 1

Means we can consider them probabilities.

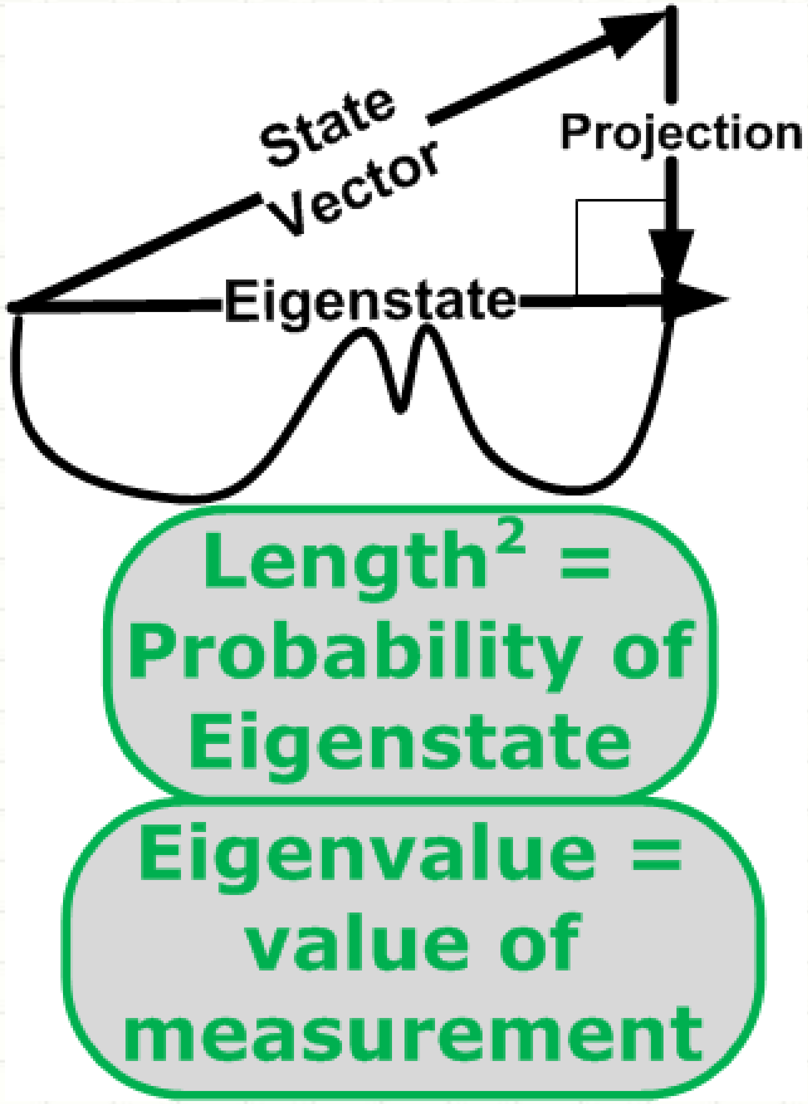
QM Background [23]

The final measured state vector of the thingie is an eigenvector of the operator.

QM Background [24]

The value of the **eigenvalue** IS the value of the measurement. ***What we read on the meter.***

QM Background [25]



QM Background [26]

SUMMARY

What	Representation
Real measureable thingie	Hermitian Operator
Unknown State of thingie	Unknown Weighted mix of Eigenvectors
Measurement of thingie	Projection onto 1 Eigenvector
Value of the Measurement	The Eigenvalue of the Eigenvector
Real Number Measurements	Hermitian Eigenvalues are Always Real
Probability of getting measured value	Squared projection (length^2)

QM Afterword 1 [27]

In Quantum Computing, why don't we see the Hermitian operators that generate the state vectors?

Because there are and will be many varied physical implementations of qubits and other objects with state. We are not primarily interested in their physical implementations. We are interested in the state vectors and the results of their manipulation, however they were generated.

In Quantum Mechanics, are there other ways to describe a system besides state vectors?

Yes. It's called a <u>Density Matrix</u>. It is introduced toward the end of the Nielsen & Chuang introduction. We may visit it later.

QM Afterword 2 [28]

What is an observable, what is a complete set of observables, what is a maximal measuremet?

Measurable quantities are also called <u>Observables</u>. If 2 observables share the same <u>eigenVECTORS</u>, they <u>commute</u>. Also they are SIMLUTANEOUSLY measurable. They may have different eigenvalues for an eigenvector. They may also have different projections on the eigenvector (probabilities).

A complete (maximal) set of observables all share the <u>same eigenvectors</u>. Their measurement is the most we can expect out of a measurememt.

QM Background [29]

Applied to Polarization

Unitary Equivalent States

$$u = \updownarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad v = \leftrightarrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\hat{u} = \text{⤡} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \hat{v} = \text{⤢} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

QM Background [30]

Applied to Polarization

$$U = \begin{pmatrix} COS\left(\frac{\pi}{4}\right) & -SIN\left(\frac{\pi}{4}\right) \\ SIN\left(\frac{\pi}{4}\right) & COS\left(\frac{\pi}{4}\right) \end{pmatrix} = \begin{pmatrix} \left(\frac{1}{\sqrt{2}}\right) & -\left(\frac{1}{\sqrt{2}}\right) \\ \left(\frac{1}{\sqrt{2}}\right) & \left(\frac{1}{\sqrt{2}}\right) \end{pmatrix}$$

$$\det U = 1,\ U\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \left(\frac{1}{\sqrt{2}}\right) \\ \left(\frac{1}{\sqrt{2}}\right) \end{pmatrix} = \left(\frac{1}{\sqrt{2}}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$U\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\left(\frac{1}{\sqrt{2}}\right) \\ \left(\frac{1}{\sqrt{2}}\right) \end{pmatrix} = \left(\frac{1}{\sqrt{2}}\right)\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Spectral definition
of
Operator H

$$H = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

QM Background [31]

Applied to Polarization

$$UHU^{-1} = \begin{pmatrix} \left(\frac{1}{\sqrt{2}}\right) & -\left(\frac{1}{\sqrt{2}}\right) \\ \left(\frac{1}{\sqrt{2}}\right) & \left(\frac{1}{\sqrt{2}}\right) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \left(\frac{1}{\sqrt{2}}\right) & \left(\frac{1}{\sqrt{2}}\right) \\ -\left(\frac{1}{\sqrt{2}}\right) & \left(\frac{1}{\sqrt{2}}\right) \end{pmatrix} =$$

$$\begin{pmatrix} \left(\frac{1}{\sqrt{2}}\right) & -\left(\frac{1}{\sqrt{2}}\right) \\ \left(\frac{1}{\sqrt{2}}\right) & \left(\frac{1}{\sqrt{2}}\right) \end{pmatrix} \begin{pmatrix} \left(\frac{1}{\sqrt{2}}\right) & \left(\frac{1}{\sqrt{2}}\right) \\ -\left(\frac{1}{\sqrt{2}}\right) & \left(\frac{1}{\sqrt{2}}\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{H}$$

$$H = \hat{H}$$

$$\begin{pmatrix} \left(\frac{1}{\sqrt{2}}\right) & -\left(\frac{1}{\sqrt{2}}\right) \\ \left(\frac{1}{\sqrt{2}}\right) & \left(\frac{1}{\sqrt{2}}\right) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \hat{v} =$$

$$\begin{pmatrix} \left(\frac{1}{\sqrt{2}}\right) & -\left(\frac{1}{\sqrt{2}}\right) \\ \left(\frac{1}{\sqrt{2}}\right) & \left(\frac{1}{\sqrt{2}}\right) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \hat{u} =$$

The operator does not change here, only the eigenvectors do.