# Functional programming

# Functional programming

- **Function evaluation** (not assignment of variables) is the basic concept for a programming paradigm that has been implemented in such **functional programming languages** as ML.

- The language ML ("Meta Language") was originally introduced in the 1970's as part of a theorem proving system, and was intended for describing and implementing proof strategies. Standard ML of New Jersey (SML) is an implementation of ML.

- The basic mode of computation in ML, as in other functional languages, is the use of the **definition** and **application** of functions (explicit and recursive).

- The basic cycle of ML activity has three parts:
  - *read* input from the user,
  - *evaluate* it, and
  - *print* the computed value (or an error message).

## Why functional programming matters?

- The key to understanding the importance of functional programming is to focus on what it adds, rather than what it takes away.

- Software becomes more and more complex. It is important to structure it well.

  Structured software is:

  - easy to write
  - easy to debug
  - easy to reuse

- Modular software is generally accepted to be the key to successful software.

  - Divide-and-conquer
  - The ways in which the original problem can be divided up depends directly on the ways in which solutions can be "glued" together.
  - New "glues" are provided in functional programming (Examples: higher-order functions, lazy evaluation, polymorphism, abstract data type).

# Applications

- Software Prototyping.
- Industrial:
  - AnnoDomini - Year 2000 remediation for Cobol
  - Shop.com Merchant System - an e-commerce database
  - Combinators for financial derivatives
- Theorem provers.
- Natural language processing and speech recognition
- Network toolkits and applications.

# General features

- The functional ascetics forbid themselves facilities which less pious programmers regard as standard.
- No re-assignment.
- No side-effects.
  - When a value is assigned it does not change during the execution of the program ⇒ **Property of referential transparency**.
  - No global variable or instance of an object.
- No explicit flow of control.
- Higher level than third generation languages.
- Construction of more reliable software ⇒ **Correctness**.

  Proof of the correctness easiest than for imperative programs.

# Plan

- Recursion (sub-chapter)
- Expressions, values and simple types
- Functions (explicit, recursive)
- Types: tuples, lists
- Operations on lists
- Pattern matching
- Higher-order functions
- Mutual-recursion
- Currying
- Scope (let, local)
- Records, arrays, user defined types
- Exceptions

# Sub-chapter: Recursion

# Defining Functions

- Functions with a **finite domain** can be described by specifying for each element in the domain the associated element in the **codomain**.

- **Examples:**
  - 
$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x = 0 \text{ or } x = 3 \end{cases}$$
  - Let $x$ a real. f(x) = 1 if $0 \leq x \leq 3$

- The two basic mechanisms for defining functions on **infinite** domains are
  - **explicit** definitions and
  - **recursive** definitions.

# Explicit definitions

- An **explicit definition** of a function $f$ consists of giving an expression that indicates for each domain element $x$ how $f(x)$ is obtained from previously defined functions (including constants) by composition.

- **Examples**

$$\begin{array}{lcl} zero(x) & = & 0 \\ add3(x) & = & x+3 \\ gt(x,y) & = & \text{if } x>y \text{ then 1 else 0} \\ \chi_A(x) & = & \text{if } x \in A \text{ then 1 else 0} \end{array}$$

*Note*

– The last function is called the *characteristic function* of the set $A$.

– If-then-else may be used for case distinctions in function definitions.

# Recursive Definitions

- A **recursive definition** of a function consists of giving an expression for every domain element $x$ that indicates how $f(x)$ is obtained from previously defined functions **and** *values of $f$ for "smaller" arguments* (by composition).

  $\rightarrow$ **Self-references**

- The **recursion principle** specifies under which conditions such definitions with self-references are **well-formed**.

- **Example**

  The number of *permutations* of $n$ elements is $n!$ (or $fact(n)$, read $n$ *factorial*).

  $\rightarrow$ **Order**

  This function can be defined recursively by:

  $$fact(n) = \quad \text{if } n = 0 \text{ then } 1 \text{ else } n * fact(n-1).$$

  The values $fact(n)$, for all $n > 0$, depend on values $fact(k)$, where $k$ is smaller than $n$. Here $k = n - 1$. This case is called the **general case**.

  $n = 0$ is called the **exit condition** or the **basis condition**.

## Well-Formed Recursive Definitions

- A **well-formed recursive definition** of a function $f$ consists of two parts:
  - the **basis case** defines the function $f$ for the "smallest" arguments in terms of previously defined functions (including constants), (*no f*).
  - the **general case** defines values $f(x)$ in terms of previously defined functions and values $f(y)$ for "smaller" arguments $y$.

- In the case of definitions of functions over the natural numbers, smaller is interpreted in the usual sense.

  Later on we will see recursive definitions of functions on other domains, such as lists, where "smaller" necessarily has to be interpreted differently. We use an **ordering** on the elements we consider.

## Computing Values of Recursively Defined Functions

- The **evaluation** of a recursively defined function for a specific argument involves two kinds of operations:
  - **substitutions** use the function definition to "expand" an application, whereas
  - **simplifications** use knowledge about previously defined (or primitive) functions to "reduce" an expression.
- The evaluation process will **terminate** if the definition is **well-formed.**
- **Example:**

$$
\begin{array}{rcll}
fact(5) & = & 5 * fact(5-1) & \text{(substitution)} \\
 & = & 5 * fact(4) & \text{(simplification)} \\
 & = & 5 * (4 * fact(4-1)) & \text{(substitution)} \\
 & = & 20 * fact(3) & \text{(simplification)} \\
 & \vdots & & \\
 & = & 120 &
\end{array}
$$

# Example: Squares

- There are different ways to define a function.
- For instance, the function that squares its argument can be defined **explicitly** in terms of multiplication,

  $$square(x) = x * x,$$

  or by **recursion**:

  $$square(x) = \text{if } x = 0 \text{ then } 0 \text{ else } square(x-1) + 2x - 1$$

  From the recursive definition we get the following function values:

  $$\begin{array}{lcl} square(0) & = & 0 \\ square(1) & = & square(0) + 1 = 1 \\ square(2) & = & square(1) + 3 = 4 \\ square(3) & = & square(2) + 5 = 9 \\ square(4) & = & square(3) + 7 = 16 \\ \vdots & & \end{array}$$

  The two definitions above define the same function, as

  $$x * x = (x-1) * (x-1) + 2x - 1.$$

## Addition and gcd

- Addition

$$add(a,b) = \begin{cases} a & \text{if } b = 0 \\ add(a, b-1) + 1 & \text{otherwise} \end{cases}$$

- Greatest Common Divisor

$$gcd(a,b) = \begin{cases} a & \text{if } b = 0 \\ gcd(b, a\ mod\ b) & \text{otherwise} \end{cases}$$

# Fibonacci Numbers

- The recursive definition of the following well-known function (*Fibonacci function*) employs the function values for **several** smaller arguments:

$$fib(n) = \begin{cases} 1 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ fib(n-1) + fib(n-2) & \text{if } n > 1 \end{cases}$$

- The corresponding function values are called **Fibonacci numbers**:

$$\begin{array}{lcccr} fib(0) & = & & & 1 \\ fib(1) & = & & & 1 \\ fib(2) & = & fib(1) + fib(0) & = & 2 \\ fib(3) & = & fib(2) + fib(1) & = & 3 \\ fib(4) & = & fib(3) + fib(2) & = & 5 \\ fib(5) & = & fib(4) + fib(3) & = & 8 \ldots \end{array}$$

- The Fibonacci numbers were originally defined to count the number of rabbits after $n$ generations, but they pop up in an amazing variety of places:
  - The *Golden Ratio* of architecture, $\phi \approx fib(n)/fib(n-1) = (1+\sqrt{5})/2 \approx 1.618$
  - The angles between leaves in spiral pine cones grow as ratios of Fibonacci numbers.
  - They arise in the analysis of computer algorithms.

# Well-defined Functions

- A key requirement of a recursive definition is that it be formulated in terms of function values for **smaller** arguments.
- A recursive function is said **well-defined**, if it is possible to compute $f(n)$ for all $n$ for which the function is defined. Otherwise it is said **partially defined**.
- Consider this definition,

$$F(x) = \text{if } x = 0 \text{ then } 0 \text{ else } F(x+1) + 1$$

and corresponding attempts at computing function values,

$$\begin{array}{rcl} F(0) & = & 0 \\ F(1) & = & F(2) + 1 \\ & = & F(3) + 2 \\ & = & F(4) + 3 \\ & \vdots & \end{array}$$

This function is defined for one argument only. So $F$ is not well-defined.

- What about the function $G$, defined for positive integers by

$$G(n) = \begin{cases} 0 & \text{if } n = 1 \\ 1 + G(n/2) & \text{if } n \text{ is even} \\ G(3n-1) & \text{if } n \text{ is odd and } n > 1 \end{cases}$$

# Study of G

- $G$ is not well-defined for all arguments.

  We have

$$
\begin{array}{lcl}
G(1) & = & 0 \\
G(2) & = & 1 + G(1) = 1 \\
G(3) & = & G(8) = 1 + G(4) = 1 + (1 + G(2)) = 3 \\
G(4) & = & 1 + G(2) = 2 \\
G(5) & = & G(14) = 1 + G(7) = 1 + G(20) \\
 & = & 1 + (1 + G(10)) = 3 + G(5) \\
\vdots & & 
\end{array}
$$

  Thus, if $G(5)$ was defined, we could infer the contradictory statement that $0 = 3$! In other words, $G(5)$ must be undefined.

# A new function $H$

- It has been *conjectured* (and shown up to one trillion) that a slight modification,

$$H(n) = \begin{cases} 0 & \text{if } n = 1 \\ 1 + H(n/2) & \text{if } n \text{ is even} \\ H(3n+1) & \text{if } n \text{ is odd and } n > 1 \end{cases}$$

  defines a well-defined function on all positive integers.

  - **H(2):** H(1)
  - **H(10):** H(5)–H(16)–H(8)–H(4)–H(2)–H(1)
  - **H(17):** H(52)–H(26)–H(13)–H(40)–H(20)–H(10)–H(5)–H(16)–H(8)–H(4)–H(2)–H(1)
  - **H(21):** H(64)–H(32)–H(16)–H(8)–H(4)–H(2)–H(1)
  - **H(35):** H(106)–H(53)–H(160)–H(80)–H(40)–H(20)–H(10)–H(5)–H(16)–H(8)–H(4)–H(2)–H(1)

  $H$ counts the number of downward steps this path takes.

  $H(2) = 1$

  $H(17) = 9$

  $H(21) = 6$

  and $H(35) = 10$.

## More General Recursive Definitions

- **Example:**

$$M(n) = \begin{cases} n - 10 & \text{if } n > 100 \\ M(M(n+11)) & \text{if } n \leq 100 \end{cases}$$

- This function is known as "McCarthy's 91 function."

  Its definition uses **nested** recursive function applications.

- Consider one instance,

$$\begin{array}{rcll} M(99) & = & M(M(110)) & (\text{since } 99 \leq 100) \\ & = & M(100) & (\text{since } 110 > 100) \\ & = & M(M(111)) & (\text{since } 100 \leq 100) \\ & = & M(101) & (\text{since } 111 > 100) \\ & = & 91 & (\text{since } 101 > 100) \end{array}$$

- Is this function defined for all arguments $n \leq 100$?

  The function is in fact defined for all positive integers and remarkably takes the value 91 for all arguments less than or equal to 101.

- M can also be defined explicitly by:

$$M(n) = \begin{cases} n - 10 & \text{if } n > 100 \\ 91 & \text{if } n \leq 100 \end{cases}$$

- Proving that the 2 definitions are equivalent requires **mathematical induction** arguments.

## Different evaluations of a recursive function

- $$f(x,y) = \begin{cases} 0 & \text{if } x = 0 \\ f(x-1, f(x,y)) & \text{otherwise} \end{cases}$$

- Consider f(1,1).

  **Innermost evaluation**

  $f(1,1) = f(0, \mathrm{f}(1,1)) = f(0, f(0, \mathrm{f}(1,1))) = \ldots$

  **Outermost evaluation**

  $f(1,1) = \mathrm{f}(0, f(1,1)) = 0$

  **Simultaneous**

  $f(1,1) = \mathrm{f}(0, \mathrm{f}(1,1)) = 0$

- Innermost evaluation does not always terminate.

- Outermost evaluation does always terminate.

- Innermost evaluation is more efficient than outermost evaluation (*Convergence*)

# Recursion – Summary

- Recursion is a general method for the definition of functions (and also a powerful technique for designing algorithms).

- Recursive definitions generally specify only partial functions (Intuitively functions not defined everywhere).

- The evaluation of recursively defined function for specific arguments is based on calculation by substitution and simplification.

- These two concepts,

  – definition by recursion and

  – evaluation by substitution and simplification,

  are the foundation of *functional programming languages* such as ML.

SML

# First SML example

- Not the "Hello World!" program!
- Here is a simple example:

```
- 3;
val it = 3 :  int
```

- The first line contains the SML prompt, followed by an expression typed in by the user and ended by a *semicolon*.
- The second line is SML's response, indicating the value of the input expression and its **type**.

# Interacting with SML

- SML has a number of **built-in** operators and **data types**.

- SML provides the standard arithmetic operators.

```
- 3+2;
val it = 5 :  int
- sqrt(2.0);
val it = 1.41421356237309 :  real
```

- The Boolean values `true` and `false` are available, as are logical operators such as `not` (negation), `andalso` (conjunction), and `orelse` (disjunction).

```
- not(true);
val it = false :  bool
- true andalso false;
val it = false :  bool
```

# Types in SML

- SML is a **strongly typed** language in that all (well-formed) expressions have a **type** that can be determined by examining the expression.

- As part of the evaluation process, SML determines the type of the output value.

  $\rightarrow$ **Inference of type**

- Simple types are:
  - `real`
  - **Examples:** $\sim 1.2$ and $1.5e12$ ($1.5 \times 10^{12}$) are reals.
  - `int`
  - **Examples:** $\sim 12$ and 14 are integers. $3 + 5$ is an integer.
  - `bool`
  - **Examples:** $true$ and $not(true)$ are booleans.
  - `string`
  - **Examples:** "nine", "" are strings.

# Binding Names to Values

- In SML one can associate identifiers with values,

```
- val three = 3;
val three = 3 :  int
```

  and thereby establish a new *value binding*,

```
- three;
val it = 3 :  int
```

- More complex *expressions* can also be used to bind values to names,

```
- val five = 3+2;
val five = 5 :  int
```

- Names can then be used in other expressions,

```
- three + five;
val it = 8 :  int
```

## Defining Functions in SML is a lot of fun!

- The general form of a function definition in SML is:

  `fun` ⟨identifier⟩ (⟨parameters⟩) = ⟨expression⟩;

- The **type** of a function is expressed using $\rightarrow$.

  It is *recursively* defined by:

  *type of the parameters* $\rightarrow$ *type of the result*

- **Example:**

  ```
  - fun double(x) = 2*x;
  val double = fn :  int -> int
  ```

  declares *double* as a function from integers to integers.

  The type of the function *double* is: `int` $\rightarrow$ `int`.

  ```
  - double(222);
  val it = 444 :  int
  ```

  The type of *double(222)* is `int`.

If we apply *double* to an argument of the wrong type, we get an error message:

```
- double(2.0);
Error:  operator and operand don't agree [tycon
mismatch]
operator domain:  int
operand:  real
in expression:
double 2.0
```

- The user may also **explicitly** specify types.

- **Example:**

```
- fun max(x:int,y:int,z:int) =
=     if ((x>y) andalso (x>z)) then x
=        else (if (y>z) then y else z);
val max = fn :  int * int * int -> int
- max(3,2,2);
val it = 3 :  int
```

The type of the function $max$ is:

$$\texttt{int * int * int} \rightarrow \texttt{int}.$$

# Recursive Definitions

- The use of recursive definitions is a main characteristic of functional programming languages.

- These languages strongly encourage the use of recursion as a structuring mechanism in preference to iterative constructs such as while-loops.

- **Example:**

```
- fun factorial(x) = if x=0 then 1
=     else x*factorial(x-1);
val factorial = fn :  int -> int
```

  The type of the function *factorial* is:

$$\texttt{int} \rightarrow \texttt{int}$$

  The definition is used by SML to evaluate applications of the function to specific arguments.

```
- factorial(5);
val it = 120 :  int
- factorial(10);
val it = 3628800 :  int
```

# Greatest Common Divisor

- The calculation of the **greatest common divisor (gcd)** of two positive integers can also be done recursively based on the following observations:
  1. $gcd(n, n) = n$,
  2. $gcd(m, n) = gcd(n, m)$, and
  3. $gcd(m, n) = gcd(m - n, n)$, if $m > n$.
- A possible definition in SML is as follows:

```
- fun gcd(m,n):int = if m=n then n
=                       else if m>n then gcd(m-n,n)
=                                    else gcd(m,n-m);

val gcd = fn :  int * int -> int

- gcd(12,30);
val it = 6 :  int
- gcd(1,20);
val it = 1 :  int
- gcd(126,2357);
val it = 1 :  int
- gcd(125,56345);
val it = 5 :  int
```

# Tuples in SML

- SML provides two ways of defining data types that represent sequences.
  - **Tuples** are *finite* sequences, where the length is arbitrary but fixed and the different components **need not be of the same type**.
  - **Lists** are finite sequences of elements of the **same type**.

- Some examples of tuples and the corresponding types are:

```
- val t1 = (1,2,3);
val t1 = (1,2,3) :  int * int * int
- val t2 = (4,(5.0,6));
val t2 = (4,(5.0,6)) :  int * (real * int)
- val t3 = (7,8.0,"nine");
val t3 = (7,8.0,"nine") :  int * real * string
```

  The type of $t1$ is `int * int * int`. The type of $t2$ is `int * (real * int)`. The type of $t3$ is `int * real * string`.

- The components of a tuple can be accessed by applying the built-in function `#i`, where $i$ is a positive number.

```
- #1(t1);
val it = 1 :  int
- #1(t2);
val it = 4 :  int
- #2(t2);
val it = (5.0,6) :  real * int
- #2(#2(t2));
val it = 6 :  int
- #3(t3);
val it = "nine" :  string
```

  If a function `#i` is applied to a tuple with fewer than $i$ components, an error results:

```
- #4(t3);
...  Error:  operator and operand don't agree
```

# Lists in SML

- Another **built-in data structure** to represent *sequences* in SML are **lists**.

- A **list** in SML is essentially a **finite** sequence of objects, all of the **same type**.

- **Examples:**

```
- [1,2,3];
val it = [1,2,3] :  int list
- [true,false, true];
val it = [true,false,true] :  bool list
- [[1,2,3],[4,5],[6]];
val it = [[1,2,3],[4,5],[6]] :  int list list
```

  The last example is a list of lists of integers, in SML notation `int list list`.

- All objects in a list must be of the same type:

```
- [1,[2]];
Error:  operator and operand don't agree
```

- The **empty list** is denoted by the following symbols:

```
- [];
val it = [] :  'a list - nil; val it = [] :
'a list
```

- Note that the type is described in terms of a *type variable* `'a`, as a list of objects of type `'a`. Instantiating the type variable, by types such as `int`, results in (different) empty lists of corresponding types.

# Operations on Lists

- SML provides some predefined functions for manipulating lists.

- The function `hd` returns the first element of its argument list.

  ```
  - hd[1,2,3];
  val it = 1 :  int
  - hd[[1,2],[3]];
  val it = [1,2] :  int list
  ```

  Applying this function to the empty list will result in an **exception** (error).

- The function `tl` removes the first element of its argument lists, and returns the remaining list.

  ```
  - tl[1,2,3];
  val it = [2,3] :  int list
  - tl[[1,2],[3]];
  val it = [[3]] :  int list list
  ```

  The application of this function to the empty list will also result in an error.

- The **types** of the two functions are as follows:

```
- hd;
val it = fn :  'a list -> 'a
- tl;
val it = fn :  'a list -> 'a list
```

# More List Operations

- Lists can be constructed by the (binary) function `::` (read *cons*) that adds its first argument to the front of the second argument.

```
- 5::[];
val it = [5] : int list
- 1::[2,3];
val it = [1,2,3] : int list
- [1,2]::[[3],[4,5,6,7]];
val it = [[1,2],[3],[4,5,6,7]] : int list list
```

  Again, the arguments must be of the right type:

```
- [1]::[2,3];
Error: operator and operand don't agree
```

- Lists can also be compared for equality:

```
- [1,2,3]=[1,2,3];
val it = true : bool
- [1,2]=[2,1];
val it = false : bool
- tl[1] = [];
val it = true : bool
```

# Defining List Functions

- **Recursion** is particularly useful for defining list processing functions.

- For example, consider the problem of defining an SML function, call it *concat*, that takes as arguments two lists of the same type and returns the concatenated list.

- What is the SML **type** of *concat*?

- For **example**, the following applications of the function *concat* should yield the indicated responses.

```
- concat([1,2],[3]);
val it = [1,2,3] :  int list
- concat([],[1,2]);
val it = [1,2] :  int list
- concat([1,2],[]);
val it = [1,2] :  int list
```

- In defining such list processing functions, it is helpful to keep in mind that a list is either
  - the empty list [] or
  - of the form *x::y*.

  The *empty list* and `::` are the constructors of the type `list`.

  For example,

```
- [1,2,3]=1::[2,3];
val it = true : bool
```

# Concatenation of Lists

- In **designing** a function for concatenating two lists *x* and *y* we thus distinguish two cases, depending on the form of *x*:
  - If *x* is an empty list, then concatenating *x* with *y* yields just *y*.
  - If *x* is of the form *x1::x2*, then concatenating *x* with *y* is a list of the form *x1::z*, where *z* is the results of concatenating *x2* with *y*. In fact we can even be more specific by observing that *x* = *hd(x)::tl(x)*.
- This suggests the following recursive definition.

```
- fun concat(x,y) = if x=[] then y
=           else hd(x)::concat(tl(x),y);
val concat = fn :  ''a list * ''a list -> ''a
list
```

- This seems to work (*at least on some examples*):

```
- concat([1,2],[3,4,5]);
val it = [1,2,3,4,5] :  int list
- concat([],[1,2]);
val it = [1,2] :  int list
- concat([1,2],[]);
val it = [1,2] :  int list
```

The result of: `concat([],[]);` is:

Warning: type vars not generalized because of
value restriction are instantiated to dummy types (X1,X2,...)
val it = [] : ?.X1 list

# More List Processing Functions

- **Recursion** often yields simple and natural definitions of functions on lists.

- The following function computes the *length* of its argument *list* by distinguishing between:
  - the empty list (the basis case) and
  - non-empty lists (the general case).

```
- fun length(L) =
=     if (L=nil) then 0
=     else 1+length(tl(L));

val length = fn :  ''a list -> int

- length[1,2,3];
val it = 3 :  int
- length[[5],[4],[3],[2,1]];
val it = 4 :  int
- length[];
val it = 0 :  int
```

- The following function has a similar recursive structure. It *doubles* all the elements in its argument list (of integers).

```
- fun doubleall(L) =
=     if L=[] then []
=     else (2*hd(L))::doubleall(tl(L));

val doubleall = fn :  int list -> int list

- doubleall[1,3,5,7];
val it = [2,6,10,14] :  int list
```

$doubleall$ is of type: $int\ list \rightarrow int\ list$. Why?

# The Reverse of a List

- **Concatenation** of lists, for which we gave a recursive definition, is actually a built-in operator in SML, denoted by the symbol **@**.

- We use this operator in the following recursive definition of a function that produces the *reverse* of a list.

```
- fun reverse(L) =
=     if L = nil then nil
=     else reverse(tl(L)) @ [hd(L)];

val reverse = fn :  ''a list -> ''a list

- reverse [1,2,3];
val it = [3,2,1] :  int list
- reverse nil;
stdIn:35.1-35.12 Warning:  type vars not generalized
because of value restriction are instantiated
to dummy types (X1,X2,...)
val it = [] :  ?.X1 list
```

# Pattern Matching

- We informally use pattern matching all the time in real life.

- Informally, a **pattern** is an expression containing **variables**, for which other expressions may be substituted. The problem of matching a pattern against a given expression consists of finding a suitable substitution that makes the pattern identical to the desired expression, if one exists at all.

- For example, we may apply the commutativity of $+$,

  $$x + y = y + x$$

  to the formula

  $$F = 1 + 2 + 3$$

  to obtain an equivalent formula

  $$3 + 2 + 1$$

  Here the "meta-variables" $x$ and $y$ were replaced by numbers. How?

# Function Definition by Patterns

- In SML there is an alternative form of defining functions via **patterns**.

- The general form of such definitions is:

```
fun  <identifier>(<pattern1>) = <expression1>
|    <identifier>(<pattern2>) = <expression2>
|    ...
|    <identifier>(<patternK>) = <expressionK>;
```

  where the identifiers, which name the function, are all the same, all patterns are of the same type, and all expressions are of the same type.

- For example, an alternative definition of the reverse function is:

```
- fun reverse(nil) = nil
= |   reverse(x::xs) = reverse(xs) @ [x];

val reverse = fn :  'a list -> 'a list
```

- In applying such a function to specific arguments, the patterns are inspected **in order** and the **first match** determines the value of the function.

## Removing Elements from Lists

- The following function removes **all** occurrences of its first argument from its second argument list.

```
- fun remove(x,L) =
=     if (L=[]) then []
=     else (if (x=hd(L))
=         then remove(x,tl(L))
=         else hd(L)::remove(x,tl(L)));

val remove = fn :  ''a * ''a list -> ''a list

- remove(1,[5,3,1]);
val it = [5,3] :  int list
- remove(2,[4,2,4,2,4,2,2]);
val it = [4,4,4] :  int list
- remove(2,nil); val it = [] :  int list
```

- We use it as an **auxiliary function** in the definition of another function that removes **all** duplicate occurrences of elements from its argument list.

```
- fun removedupl(L) =
=     if (L=[]) then []
=     else hd(L)::remove(hd(L),removedupl(tl(L)));

val removedupl = fn :  ''a list -> ''a list
```

# Constructing Sublists

- A **sublist** of a list $L$ is any list obtained by deleting some (i.e., zero or more) elements from $L$.

- For example, `[]`, `[1]`, `[2]`, and `[1,2]` are all the sublists of `[1,2]`.

- Let us **design** an SML function that constructs **all** sublists of a given list $L$. The definition will be **recursive**, based on a case distinction as to whether $L$ is the empty list or not.

- If $L$ is non-empty, it has a first element $x$. There are two kinds of sublists: those containing $x$, and those not containing $x$.

- For instance, in the above example we have sublists `[1]` and `[1,2]` on the one hand, and `[]` and `[2]` on the other hand.

- Note that there is a one-to-one correspondence between the two kinds of sublists, and that each sublist of the latter kind is also a sublist of `tl(L)`.

## Constructing Sublists (continued)

- These observations lead to the following definition.

```
- fun sublists(L) =
=     if (L=[]) then [nil]
=     else sublists(tl(L))
=             @ insertL(hd(L),sublists(tl(L)));

val sublists = fn :  ''a list -> ''a list list

- sublists[];
stdIn:84.1-84.11 Warning:  type vars not generalized
because of value restriction are instantiated
to dummy types (X1,X2,...)
val it = [[]] :  ?.X1 list list - sublists[1,2];
val it = [[],[2],[1],[1,2]] :  int list list
- sublists[1,2,3];
val it = [[],[3],[2],[2,3],[1],[1,3],[1,2],[1,2,3]]
:  int list list
- sublists[4,3,2,1];
val it = [[],[1],[2],[2,1],[3],[3,1],[3,2],
[3,2,1],[4],[4,1],...
```

- Here **@** denotes the (built-in) concatenation operation on lists, and the function `insertL` inserts its first argument at the front of all elements in its second argument (which must be a list). Its definition is left as an exercise.

- If we change the expression in the else-branch to

```
= else insertL(hd(L),sublists(tl(L)))
=        @ sublists(tl(L))
```

all sublists will still be generated, but in a different order.

# Higher-Order Functions

- In functional programming languages, parameters may denote **functions** and be used in definitions of other, so-called **higher-order**, functions.

- One example of a higher-order function is the function `apply` defined below, which applies its first argument (a function) to all elements in its second argument (a list of suitable type).

```
- fun apply(f,L) =
=     if (L=[]) then []
=     else f(hd(L))::(apply(f,tl(L)));
val apply = fn :  (''a -> 'b) * ''a list ->
'b list
```

  We may apply `apply` with any function as argument.

```
- fun square(x) = (x:int)*x;
val square = fn :  int -> int
- apply(square,[2,3,4]);
val it = [4,9,16] :  int list
```

- The function `doubleall` we defined may be considered a special case of supplying `apply` with first argument `double` (a function we defined in a previous lecture).

```
- apply(double,[1,3,5,7]);
val it = [2,6,10,14] :  int list
```

- `apply` is predefined in SML and is called `map`.

# Mutual Recursion

- Sometimes the most convenient way of defining (two or more different) functions is in **mutual dependence of each other**.

- Consider the functions, `even` and `odd` that test if a number is even and odd. We can define them in the following way.

```
- fun even(0) = true
= | even(m) = odd(n-1)
=  and
=  odd(0) = false
=  | odd(n) = even(n-1);
val even = fn :  int -> bool
val odd = fn :  int -> bool
```

  SML uses the keyword `and` (not to be confused with the logical operator `andalso`) for such mutually recursive definitions.

  Neither of the two definition is acceptable by itself.

```
- even(2);
val it = true :  bool
- odd(3);
val it = true :  bool
```

- Consider two functions, `take` and `skip`, both of which extract alternate elements from a given list, with the difference that `take` starts with the first element (and hence extracts all elements at odd-numbered positions), whereas `skip` skips the first element (and hence extracts all elements at even-numbered positions, if any).

```
- fun take(L) =
=     if L = nil then nil
=     else hd(L)::skip(tl(L))
= and
=     skip(L) =
=     if L=nil then nil
=     else take(tl(L));
val take = fn :  ''a list -> ''a list
val skip = fn :  ''a list -> ''a list
```

```
- take[1,2,3];
val it = [1,3] :  int list
- skip[1,2,3];
val it = [2] :  int list
```

# Sorting

- We next design a function for **sorting a list of integers**.
- More precisely, we want to define an SML function,

  ```
  sort :  int list -> int list
  ```

  such that `sort(L)` is a sorted version (in non-descending order) of `L`.
- Sorting is an important problem for which a large variety of different algorithms have been proposed.
- The method we will explore is based on the following idea. To sort a list `L`,
  - first **split** `L` into two disjoint sublists (of about equal size),
  - then (recursively) **sort** the sublists, and
  - finally **merge** the (now sorted) sublists.

  This recursive method is known as **Merge-Sort**.
- It evidently requires us to define suitable functions for
  - splitting a list into two sublists and
  - merging two sorted lists into one sorted list.

# Merging

- First we consider the problem of merging two sorted lists.
- A corresponding recursive definition can be easily defined by distinguishing between the different cases, as to whether one of the argument lists is empty or not.
- The following SML definition is formulated in terms of **patterns** (against which specific arguments in applications of the function will be matched during evaluation).

```
- fun merge([],M) = M
= |  merge(L,[]) = L
= |  merge(x::xl,y::yl) =
=     if (x:int)<y then x::merge(xl,y::yl)
=          else y::merge(x::xl,yl);
val merge = fn :  int list * int list -> int
list
- merge([1,5,7,9],[2,3,5,5,10]);
val it = [1,2,3,5,5,5,7,9,10] :  int list
- merge([],[1,2]);
val it = [1,2] :  int list
- merge([1,2],[]);
val it = [1,2] :  int list
```

- How do we split a list? Recursion seems to be of little help for this task, but fortunately we have already defined suitable functions that solve the problem.

# Merge Sort

- Using `take` and `skip` to split a list, we obtain the following function for sorting.

```
- fun sort(L) =
=     if L=[] then []
=     else merge(sort(take(L)),sort(skip(L)));
val sort = fn :  int list -> int list
```

  Don't run this function, though, as it doesn't quite work. Why?

- To see where the problem is, observe what the result is of applying `take` to a one-element list.

```
- take[1];
val it = [1] :  int list
```

  Thus in this case, the first recursive call to `sort` will be applied to the same argument!

- Here is a modified version in which one-element lists are dealt with correctly.

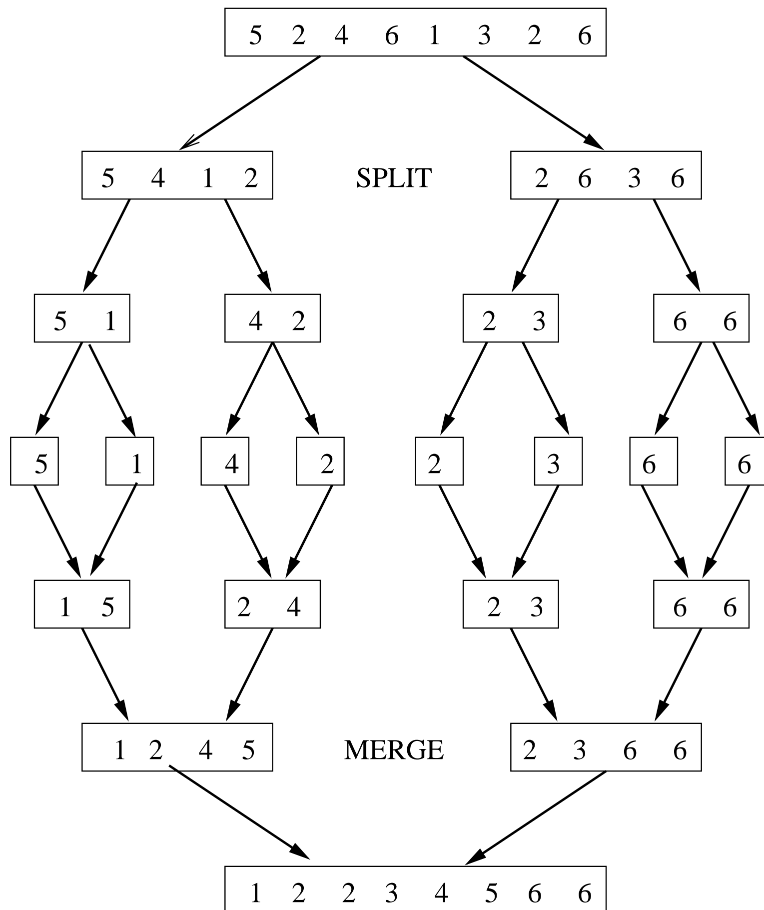
```
- fun sort(L) =
=     if L=[] then []
=     else if tl(L)=[] then L
=     else merge(sort(take(L)),sort(skip(L)));
val sort = fn :  int list -> int list
```

Finally, some examples:

```
- sort[];
val it = [] :  int list
- sort[1];
val it = [1] :  int list
- sort[1,2];
val it = [1,2] :  int list
- sort[2,1];
val it = [1,2] :  int list
- sort[1,2,3,4,5,6,7,8,9];
val it = [1,2,3,4,5,6,7,8,9] :  int list
- sort[9,8,7,6,5,4,3,2,1];
val it = [1,2,3,4,5,6,7,8,9] :  int list
- sort[1,2,1,2,2,1,2,1,2,1];
val it = [1,1,1,1,1,2,2,2,2,2] :  int list
```

# Tracing Mergesort

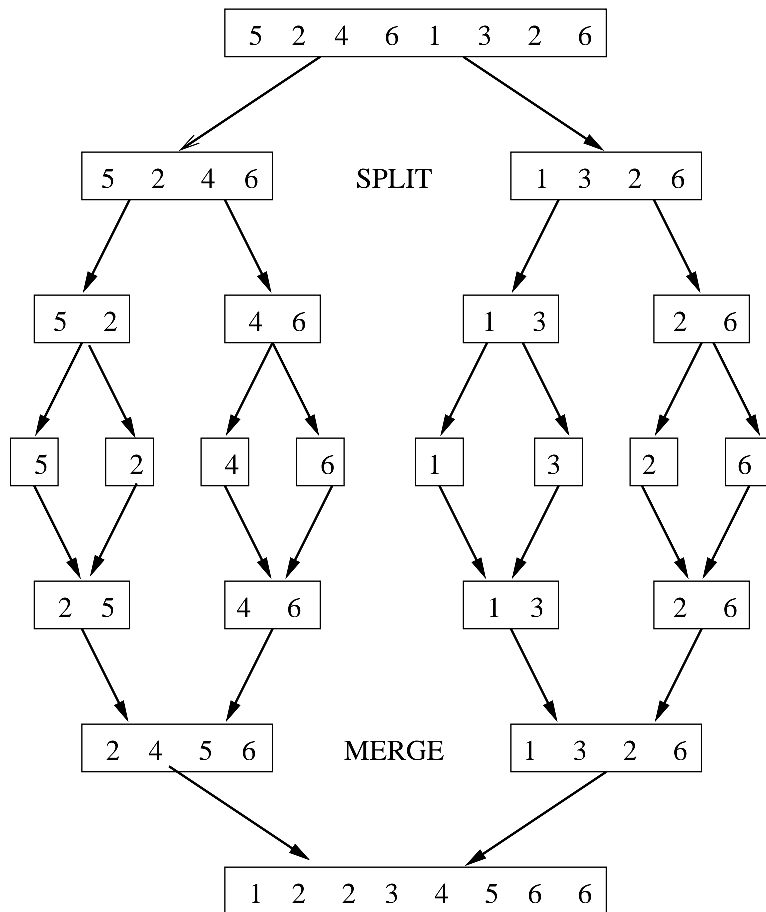
- It is important to be able to trace the execution of the mergesort program to convince yourself that it works correctly.




- In the course of executing the recursive algorithm, the computer has to keep track of what work still needs to be done as it is interrupted with additional recursive calls.

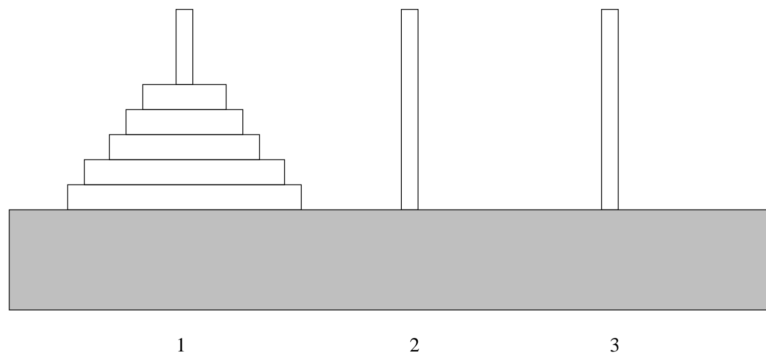
# Tracing Mergesort

How to split?

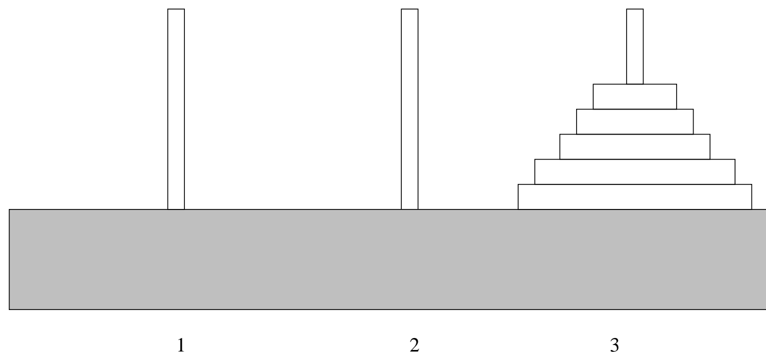


# The Tower of Hanoi

- The **tower of Hanoi** consists of a fixed number of **disks** stacked on a pole in decreasing size, that is, with the smallest disk at the top.



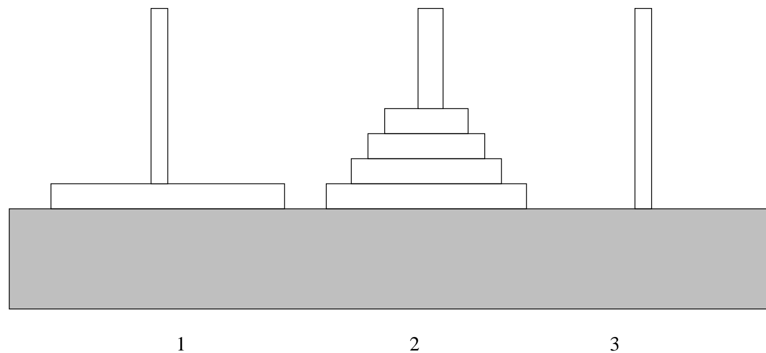

- There are two other **poles** and the task is to transfer all disks from the first to the third pole, one at a time without ever placing a larger disk on top of a smaller one.




- There is an elegant solution to this problem by recursion.

# Tower Moves

- First consider how many moves are needed, at the least, to transfer a tower of $k$ disks.
- Observe that we need to get to the following intermediate configuration, so as to be able to move the largest disk.




That is, we have to transfer the $k-1$ smaller disks to the middle pole, we can then move the largest disks from the first to the third pole, and finally the $k-1$ smaller disks from the second pole to the third pole.

- Let $M(k)$ be the minimum number of moves required to transfer $k$ disks from one pole to another pole. This function $M$ satisfies the recursive identity:

$$M(k) = M(k-1) + 1 + M(k-1) = 2M(k-1) + 1,$$

for all $k > 0$.
In addition, we set $M(0) = 0$, so that by the above identity $M(1) = 1$, which is correct as one move suffices to transfer a tower containing only a single disk.

# Minimum Number of Moves

- M(0) = 0
  M(k) = M(k-1) + 1 + M(k-1) = 2M(k-1) + 1 for all $k > 0$.

- Let us evaluate the function for some arguments:

$$\begin{aligned}
M(0) &= 0 \\
M(1) &= 2M(0) + 1 = 1 \\
M(2) &= 2M(1) + 1 = 3 \\
M(3) &= 2M(2) + 1 = 7 \\
M(4) &= 2M(3) + 1 = 15 \\
M(5) &= 2M(4) + 1 = 31 \\
M(6) &= 2M(5) + 1 = 63 \\
&\vdots
\end{aligned}$$

- The values grow fairly fast. In fact one can show that the function $M$ can be explicitly defined by

$$M(k) = 2^k - 1,$$

  for all $k \geq 0$. That is, function values grow exponentially with the argument.

- This tells us that a lot of moves are needed to transfer a tall tower, though we don't know the actual sequence of moves yet. For that purpose we will write an SML function.

# Tower of Hanoi in SML

- Poles are represented by the numbers 1, 2, and 3.

- We represent a **move** as a pair of integers $(x, y)$. That is, $(x, y)$ is interpreted as moving a disk from pole $x$ to pole $y$.

  The pair `(x,y)` is an example of a tuple of length 2 and type `int * int`.

- The function `ttower` takes three integer arguments $k$, $x$, and $y$ such that $k \geq 0$, $1 \leq x \leq 3$ and $1 \leq y \leq 3$. It returns a **list of moves** that transfer a tower of $k$ discs from pole $x$ to pole $y$.

  The result returned by `ttower` is of type `(int * int) list`.

- The function `comp`, if provided with two of the numbers 1, 2, or 3 as arguments (2 poles), returns the third (pole).

```
- fun comp(x,y) = 6-(x+y);
val comp = fn :  int * int -> int
- comp(3,1);
val it = 2 :  int
```

- The function `ttower` is defined by:

```
- fun ttower(k,x,y) =
=     if (k=0 orelse x=y) then []
=     else if k=1 then [(x,y)]
=     else ttower(k-1,x,comp(x,y))
=             @ ((x,y)::ttower(k-1,comp(x,y),y));
val ttower = fn :  int * int * int
.                      -> (int * int) list
```

– The second line indicates that no move is needed if $k = 0$ or the tower is to remain at the same pole.

– The third line provides an explicit solution for moving a tower of one disk.

– The fourth and fifth line show that in the general case we can

(a) move $k-1$ disks from $x$ to the "auxiliary" pole $z$,
(b) move the largest disk from $x$ to $y$, and
(c) move $k-1$ disks from $z$ to $y$.

- Here are some simple sequences of moves,

```
- ttower(1,1,3);
val it = [(1,3)] :  (int * int) list

- ttower(2,2,2);
val it = [] :  (int * int) list
```

  and a few longer ones,

```
- ttower(2,1,3);
val it = [(1,2),(1,3),(2,3)] :  (int * int)
list

- ttower(3,1,3);
val it = [(1,3),(1,2),(3,2),(1,3),(2,1),(2,3),(1,3)]
:  (int * int) list

- ttower(4,1,3);
val it = [(1,2),(1,3),(2,3),(1,2),(3,1),
.          (3,2),(1,2),(1,3),(2,3),(2,1),
.          (3,1),(2,3),(1,2),(1,3),(2,3)]
:  (int * int) list
```