Polymorphism

- The ability of a function to allow **arguments** of different types is called **polymorphism**.

  Examples:

  – $hd\ :\ 'a\ list \rightarrow\ 'a$.

  – fun id(x) = x; $id\ :\ 'a \rightarrow\ 'a$

  What is the type of (id, id)?

- Default type.

  Example: $fun\ f(x)\ =\ x + x$; f is of type $int \rightarrow int$.

- Functions that restricts polymorphism:

  – $+$, $-$, $*$, $\sim$

  – $/$, div, mod

  – $<$, $<=$, $>=$, $>$

  – $andalso$, $orelse$, not

  – string concatenation.

  – Conversion type operators: ord, chr, $real$, str, $floor$, $ceiling$, $round$, $truncate$

- Functions allowing the polymorphism:

  – () (for tuples), #1, #2

  – ::, @, hd, tl, nil, [] (for lists)

  – =, <>.

Equality operators

- Type variables whose values are restricted to be an **equality type** are distinguished by having names that begin with two quote marks rather then only one ($''a$).

- Examples:

```
(1,2) = (2,3);
val it = false: bool
[1,2] = [1,2,3];
val it = false: bool
[1,2] <> [1,2,3];
val it = true: bool
1.2 = 1.2;
stdIn:1.1-2.5 Error: operator and operand don't agree
[equality type required]
operator domain: ''Z * ''Z
operand: real * real
in expression:
1.2 = 1.2
```

Pattern matching and equality operators

- Pattern matching permits us to design more general functions.

- Example: Can we use the function $reverse$ (previously defined) to reverse a list of reals?

```
fun reverse(L) =
if L = nil then nil
else reverse(tl(L)) @ [hd(L)];
val reverse = fn : ''a list -> ''a list
```

$reverse$ cannot be used. But $reverse1$ can be used thanks to pattern matching.

```
fun reverse1(nil) = nil
reverse1(x::L) = reverse(L) @ [x];
val reverse1 = fn : 'a list -> 'a list
```

Curried functions

- Until now we defined functions having only one parameter that is a tuple. A **curried function** is a function having several parameters (a list of parameters with no parentheses or commas).
- Example 1:

  map (built-in function in SML) is the curried version of $apply$. The type of map is $(''a->'b)->''alist->'blist$.

```
fun apply(f,L) =
if (L=[]) then []
else f(hd(L))::(apply(f,tl(L)));
val apply = fn : (''a -> 'b) * ''a list -> 'b list
```

- Example 2:

  Let us consider the function exp that computes x^y (x is a real and y is an int). $exp1$ is the curried form of exp.

```
fun exp(x,0) = 1.0
| exp(x,y) = x * exp(x,y-1);
val exp: fn: real * int -> real
fun exp1 x 0 = 1.0
| exp1 x y = x * exp1 x y-1;
val exp1: fn: real -> int -> real
```

Partially instantiated functions

- Curried functions are useful to construct new functions by applying the function to arguments for some but not all of its parameters.
- Examples:

  $exp1$ 3.0 is a function of type $int->real$ that computes the powers of 3.0.

  $exp1$ 10.0 is a function of type $int->real$ that computes the powers of 10.0.

Exceptions

- Many functions are *partial*, they do not produce a value for some of the possible arguments of the function's domain type. It is essential that we be able to catch such errors. This is done by generations and handling of exceptions.
- Examples of predefined exceptions in SML:

```
5 div 0;
uncaught exception Div


hd(nil);
uncaught exception Empty
```

Exceptions

- **Declaration of an exception**
- Syntax:

  $exception\ <name\ of\ the\ exception>\ (of\ <type>)$
- Examples:

```
exception Foo;
exception Alert of string;
exception Number of int;
exception Tuple of int*real;
```

- The type of an exception is exn.
- **Raising an exception**
- Syntax:

  $raise\ <name\ of\ the\ exception(<parameters>)>$
- Examples:

```
raise Foo
raise Alert(``Stop'')
raise Number(5)
raise Tuple(1,5)
```

- **Handling an exception**
- Syntax:

  $< expression >$ $handle$ $< match >$
- Examples:

  Design a function $g(n, m)$ that prints "Division by 0" in case m = 0 using exceptions.

```
exception Division of int;
exception Division of int

fun f(n,m) =
if m = 0
then raise Division(0)
else n div m;
val f = fn : int * int -> int

fun g(n,m) = f(n,m) handle
Division(0) => (print "Division by 0";
print "n";
0);
val g = fn : int * int -> int

g(1,0);
Division by 0
val it = 0 : int
g(2,1);
val it = 2 : int
```

Variables and environment

- A variable is represented by a name and a value.
- The environment is a list of pairs $(variable, value)$.

  $\rho = \{(v, val) \mid v \textit{ is a variable and } val \in Value(v)\}$
- A new variable may be added to the environment using:

```
val <variable> = <value>;
```

- Example:

  Initial environment: ρ_0

  `val x = 1;`

  `val y = 3;`

  `val z = y+x+3;`

  Environment: $\rho_1 = \rho_0 \cup \{(x, 1),\ (y, 3),\ (z, 7)\}$

  `val y = 1.2;`

  Environment: $\rho_2 = \rho_1 \cup \{(y, 1.2)\}$
- View of the environment.
- Example: $\rho_2 = \rho_0 \cup \{(x, 1),\ (y, 1.2),\ (z, 7)\}$

Local environment

- It is possible to create local variables inside a function using *let ... in ... end*.
- Syntax:

```
let
val <var1> = <val1>;
val <var2> = <val2>;
...
val <varn> = <valn>
in
<expression>
end
```

- Example:

  $\rho_1 = \{(x, 1),\ (y, 2),\ (z, 3)\}$

```
let val u=x+y; val v=x*y+x;val z=1
in u*z+v+x
end;
val it = 9 : int
u;
stdIn:160.1 Error: unbound variable or constructor:
u
```

  $\rho_2 = \rho_1 \cup \{(u, 3),\ (v, 5),\ (z, 1)\}$

Functions

- One can use let in expressions to allow us to use common subexpressions.
- Examples:

```
fun hundredthpower(x:real) =
let val four = x*x*x*x;
val twenty = four*four*four*four
in
twenty*twenty*twenty*twenty*twenty
end;
val hundredthpower = fn : real -> real

fun fib(n)=
if ((n = 0) orelse (n = 1)) then 1
else
let
val n1 = fib(n-1);
val n2 = fib(n-2)
in n1 + n2 end;
```

Pattern matching

- In SML it is possible to split apart the values returned by a function.
- Example:

```
fun f(x) = (x,x,x);
val f = fn : 'a -> 'a * 'a * 'a

val (a,b,c) = f(3);
val a = 3 : int
val b = 3 : int
val c = 3 : int
The variable a is linked with 3, the variable b is
linked with 3 and the variable c is linked with 3.
val t = f(3);

val t = (3,3,3) : int * int * int
```

- This can be used in functions.

```
fun g(nil) = 0
| g(L) =
let
val (a,b,c) = f(hd(L)*2)
in a
end;
val g = fn : int list -> int

g(nil);
val it = 0 : int
g([4,6,7]);
val it = 8 : int
```

User defined types

- We can build new types in ML using “old” types using:
  - the “product of types” T1*T2
  - function types: $T1 \to T2$
  - *list*

  Examples:

```
type intpair = int * int;
(5,7);
val it = (5,7) : int * int
(5,7): intpair;
val it = (5,7) : intpair
type funonint = int -> int;
type intlist = int list;
```

- Parametrized type definitions

  Syntax:

```
type (<list of parameters>) <identifier> =
<type expression>;
```

  Examples:

```
type ('a,'b) relation = ('a * 'b) list;
val R = [(1,2),(2,3)]:(int,int)relation;
```

Datatypes

- SML has also a very powerful mechanism for defining new types called **datatypes**.
- A datatype definition involves:
  - A **type constructor** that is the name of the datatype.
  - One or more **data constructors**, which are identifiers used as operators to build values belonging to a new datatype.
- Functions having parameters of user defined datatypes are generally defined easily using **pattern matching**.
- Syntax:

```
datatyoe (<list of type parameters>) <identifier>
=
<first constructor expression> |
<second constructor expression> |
...
<last constructor expression>;
```

Enumerated types

- An **enumerated datatype** declaration consists of the keyword **datatype**, a name for the datatype, and a list of data constructors separated by |.
- Examples:

```
datatype primary_color = Red | Blue | Green;
datatype primary_color = Red | Blue | Green
datatype printer_color = Yellow | Magenta | Cyan;
datatype printer_color = Yellow | Magenta | Cyan
datatype fruit = Apple | Pear | Grape;
datatype fruit = Apple | Grape | Pear
datatype bin = zero | one;
datatype bin = one | zero
fun isApple(x) = (x=Apple);
val isApple = fn : fruit -> bool
```

Constructors in datatype definitions and functions

- Example:

```
datatype Nat = Zero | Suc of Nat;
```

The constructors are *Zero* and *Suc*. The constructor *Suc* takes one parameter. *Zero* does not take parameters. This datatype is *recursive*: Nat is used in the rhs of its own definition.

- Define the Addition of 2 Nats, the multiplication of 2 Nat, the conversion from a Nat to an int and from an int to a Nat (curried forms).

```
fun plus n Zero = n
| plus n (Suc(p)) = Suc(plus n p);
val plus = fn : Nat -> Nat -> Nat
plus (Suc(Zero)) (Suc(Zero))
val it = Suc (Suc Zero) : Nat

fun times n Zero = Zero
| times n (Suc(p)) = plus (times n p) n ;
val times = fn : Nat -> Nat -> Nat
times (Suc(Zero)) (Suc(Zero))
val it = Suc Zero : Nat
```

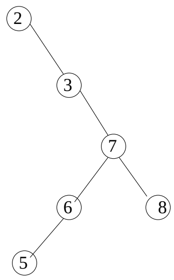
```
fun natint Zero = 0
| natint (Suc(n)) = natint(n) + 1;
val natint = fn : Nat -> int
natint(Suc(Suc(Suc(Zero))));
val it = 3 : int

fun intnat 0 = Zero
| intnat n = Suc(intnat(n-1));
val intnat = fn : int -> Nat
intnat(3);
val it = Suc (Suc (Suc Zero)) : Nat
```

Binary Search Trees

A binary search tree is a binary tree where each node contains a key such that:

- All keys in the left subtree precede the key in the root.
- All keys in the right subtree succeed the key in the root.
- The left and right subtrees of the root are again binary search trees.



Binary Search Trees in SML

The *datatype* constructor in SML permits us to define new data types, like a BinaryTree:

```
- datatype 'a BinaryTree = btempty |
=  bt of 'a * 'a BinaryTree * 'a BinaryTree ;
datatype 'a BinaryTree
con bt :  'a * 'a BinaryTree * 'a BinaryTree ->
'a BinaryTree
con btempty :  'a BinaryTree
```

Each BinaryTree consists of a *header* bt, a node value or *key*, and left / right subtrees, both of which must be BinaryTrees.

The special case of a tree with no keys in it, the empty tree, is denoted $btempty$.

```
- val Tree = bt(2,btempty,
=              bt(3,btempty,
=                   bt(7,bt(6,bt(5,btempty,btempty),
=                           btempty),
=                        bt(8,btempty,btempty))
=              )
=           );
= val Tree = bt (2,btempty,bt (3,btempty,bt #)) :  int
BinaryTree
```

The lookup Function

To look up an element in a binary search tree, we must test the label of the root against our desired key, and the recursively search the appropriate side's subtree:

```
- fun lookup (btempty,_) = false
= |   lookup(bt(root:int,left,right),x:int) =
=      if (x = root) then true
=      else (if (x <= root) then lookup(left,x)
=           else lookup(right,x) );
= val lookup = fn :  int BinaryTree * int -> bool
```

The search ends in failure whenever the subtree is $btempty$.

```
- lookup(Tree,6);
val it = true :  bool
- lookup(Tree,1);
val it = false :  bool
- lookup(Tree,9);
val it = false :  bool
- lookup(Tree,8);
val it = true :  bool
- lookup(btempty,6);
val it = false :  bool
```

Tree Traversal in ML

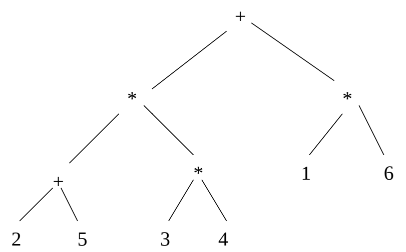
Note how only the order of the recursive calls changes in the three traversals.

```
fun inorder (bt_empty) = []
|  inorder(bt(root:'a,left,right)) =
inorder(left) @ (root ::  inorder(right));

fun preorder (bt_empty) = []
|  preorder(bt(root:'a,left,right)) =
root ::  (preorder(left) @ preorder(right));

fun postorder (bt_empty) = []
|  postorder(bt(root:'a,left,right)) =
(postorder(left) @ postorder(right)) @ (root ::
[]);
```

Example



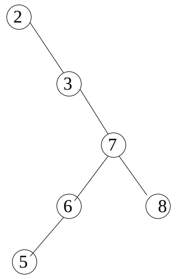

```
- val Expression =
= bt("+",
-   bt("*",
-       bt("+",
-          bt("2",bt_empty,bt_empty),
-          bt("5",bt_empty,bt_empty) ),
-       bt("*",
-          bt("3",bt_empty,bt_empty),
-          bt("4",bt_empty,bt_empty) ) ),
-    bt("*",
-       bt("1",bt_empty,bt_empty),
-       bt("6",bt_empty,bt_empty) ) );
= val Expression = bt ("+",bt ("*",bt #,bt #),bt
("*",bt #,bt #)) :  string BinaryTree
```

```
- inorder(Expression);
val it = ["2","+","5","*","3","*","4","+","1","*","6"]
-
- preorder(Expression);
val it = ["+","*","+","2","5","*","3","4","*","1","6"]
-
- postorder(Expression);
val it = ["2","5","+","3","4","*","*","1","6","*","+"]
```

These traversal functions could be easily modified to compute the value of the arithmetic tree instead of just returning the formula.

Inorder Traversals of Binary Search Trees

Why is an in-order traversal called in order?



Note that an in-order traversal of a binary search tree lists all the keys in alphabetical order:

```
- inorder(Tree);
val it = [2,3,5,6,7,8] :  int list
```

(* Binary tree processing *)

```
datatype 'a BinaryTree = bt_empty |
bt of 'a * 'a BinaryTree * 'a BinaryTree ;

fun left_subtree bt_empty = bt_empty
| left_subtree(bt(_,left,_)) = left;

fun right_subtree bt_empty = bt_empty
| right_subtree(bt(_,_,right)) = right;

exception label_has_nil_argument;

fun label bt_empty = raise label_has_nil_argument
| label(bt(value,_,_)) = value;
```

(* Sample binary trees *)

```
val Tree = bt(2,bt_empty,
bt(3,bt_empty,
bt(7,bt(6,bt(5,bt_empty,bt_empty),
bt_empty),
bt(8,bt_empty,bt_empty))
)
);

val Tree1 = bt(3,bt_empty,bt_empty);

val Tree2 = bt(5,bt(1,bt_empty,bt_empty),bt_empty);

val Tree3 = bt(7,bt(4,bt_empty,bt_empty),
bt(12,bt_empty,bt_empty));

val Tree4 = bt("*",
bt("/",
bt("-",bt("7",bt_empty,bt_empty),
bt("a",bt_empty,bt_empty) ),
bt("5",bt_empty,bt_empty) ),
bt("exp",
bt("+",bt("a",bt_empty,bt_empty),
bt("b",bt_empty,bt_empty) ),
bt("3",bt_empty,bt_empty) ));

val Expression = bt("+",
bt("*",
bt("+",
bt("2",bt_empty,bt_empty),
bt("5",bt_empty,bt_empty) ),
bt("*",
bt("3",bt_empty,bt_empty),
bt("4",bt_empty,bt_empty) ) ),
bt("*",
bt("1",bt_empty,bt_empty),
bt("6",bt_empty,bt_empty) ) );

lookup(Tree,6);
lookup(Tree,1);
lookup(Tree,8);
lookup(Tree,9);
lookup(bt_empty,6);
```

```
inorder(Tree);
preorder(Tree);
postorder(Tree);
inorder(Expression);
preorder(Expression);
postorder(Expression);
```

Arrays

- Array is a structure in SML that gives us the ability to create and manipulate arrays.

  The structure is opened using:

```
open Array;
```

- opening Array

```
type 'a array = 'a ?.array
type 'a vector = 'a ?.vector
val maxLen : int
val array : int * 'a -> 'a array
val tabulate : int * (int -> 'a) -> 'a array
val fromList : 'a list -> 'a array
val length : 'a array -> int
val sub : 'a array * int -> 'a
val update : 'a array * int * 'a -> unit
val extract : 'a array * int * int option -> 'a vector
val copy: di:int, dst:'a array, len:int option, si:int,
src:'a array
-> unit
val copyVec:di:int, dst:'a array,len:int option,si:int,
src:'a vector
->unit
val app : ('a -> unit) -> 'a array -> unit
val foldl : ('a * 'b -> 'b) -> 'b -> 'a array -> 'b
val foldr : ('a * 'b -> 'b) -> 'b -> 'a array -> 'b
```

```
val modify : ('a -> 'a) -> 'a array -> unit
val appi : (int * 'a -> unit) -> 'a array * int *
int option -> unit
val foldli : (int * 'a * 'b -> 'b) -> 'b -> 'a array
* int * int option -> 'b
val foldri : (int * 'a * 'b -> 'b) -> 'b -> 'a array
* int * int option -> 'b
val modifyi : (int * 'a -> 'a) -> 'a array * int *
int option -> unit
```

- Example:

```
val A = array(10,4);
val A = [|4,4,4,4,4,4,4,4,4,4|] : int array
val l = length(A);
val l = 10 : int
val v = sub(A,3);
val v = 4 : int
update(A,5,0);
A;
val it = [|4,4,4,4,4,0,4,4,4,4|] : int array
```

Records

- Aggregate structures, where each component has a name.

- Example:

```
type king = {name : string, born : int,
crowned : int,died : int};

val HenryV = {
name = "Henri",
born =1387,
crowned=1413,
died = 1422
};
```

- Accessing components in a record:

  – by pattern matching:

```
val {born = x , ...} = HenryV;
val x = 1387:int
```

  - by operators:

```
val byear = #born(HenryV);
val byear = 1387 : int
```

- Functions over records:

```
fun livetime(k:king) = (#died(k)) - (#born(k));
val livetime = fn : king -> int
livetime(HenryV);
val it = 35 : int
```